NUSC Technical Report 8667
31 January 1990

Evaluation of Attenuation/Minimum-Phase Pairs By Means of Two Fast Fourier Transforms

by
Albert H. Nuttall
Surface ASW Directorate

Naval Underwater Systems Center
Newport, Rhode Island/New London, Connecticut

Preface

This research was conducted under NUSC Project No. A75215, Subproject No. R00N000, "Determination of Concentrated Energy Distribution Functions in the Time-Frequency Plane," Principal Investigator Dr. Albert H. Nuttall (Code 304). This technical report was prepared with funds provided by the NUSC In-House Independent Research and Independent Exploratory Development Program, sponsored by the Office of the Chief of Naval Research.

The technical reviewer for this report was James M. Powers (Code 2131).

Reviewed and Approved: 31 January 1990

Edward J. Sullivan / for

Daniel M. Viccione
Associate Technical Director
Research and Technology

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
31 January 1990

3. REPORT TYPE AND DATES COVERED
Progress

4. TITLE AND SUBTITLE
EVALUATION OF ATTENUATION/MINIMUM-PHASE PAIRS BY MEANS OF TWO FAST FOURIER TRANSFORMS

5. FUNDING NUMBERS
PE 61152N

6. AUTHOR(S)
Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Underwater Systems Center
New London, Connecticut 06320

8. PERFORMING ORGANIZATION REPORT NUMBER
TR 8667

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of the Chief of Naval Research
Arlington, Virginia 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

A numerically efficient method of obtaining the minimum-phase characteristic corresponding to a measured attenuation (or decibel gain) response of a linear network, by means of two fast Fourier transforms, is presented and programmed in BASIC. A method of extrapolating the measured attenuation to very small and large frequencies, as required by the theoretical transformations, is suggested. The attendant logarithmic singularities in the attenuation are subtracted and handled separately, leaving a residual that is well behaved for numerical Fourier transformation.

14. SUBJECT TERMS
Analytic Transfer Function
Attenuation
Causal Impulse Response
Decibel Gain
Fast Fourier Transform
Fourier Transform

15. NUMBER OF PAGES
66

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT
UNCLASSIFIED

18. SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

19. SECURITY CLASSIFICATION OF ABSTRACT
UNCLASSIFIED

20. LIMITATION OF ABSTRACT
UL

14. Subject Terms (Cont'd.)

Hilbert Transform
Logarithmic Singularities
Minimum-Phase Filter
Phase
Poles and Zeros
Principal Value Integral

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

τ	time delay, (1)
$h(\tau)$	impulse response, (1)
f	frequency, (1)
$H(f)$	transfer function, (1)
$\underline{F}$	Fourier transform, (1)
sub r	real part, (2)
sub i	imaginary part, (2)
sub e	even part, (4),(5)
sub o	odd part, (4),(5)
$\underline{H}$	Hilbert transform, (8)
$\fint$	principal value integral, (8),(A-1)
$\oplus$	convolution, (8)
$U(x)$	unit step function, (9)
$\delta(f)$	delta function, (10)
$\underline{h}(\tau)$	auxiliary function, (19),(20)
$\underline{F}^{-1}$	inverse Fourier transform, (19)
$b_H(\tau)$	Hilbert transform of $b(\tau)$, (28)
$B(f)$	spectrum of $b(\tau)$, (29)
C_1, C_2	contours in complex f-plane, (36),figure 1
$Q(f)$	auxiliary function, (38)
$q(\tau)$	inverse Fourier transform of $Q(f)$, (39)
$\alpha(f)$	attenuation, (42)
$\beta(f)$	phase shift, (42)
$\underline{g}(\tau)$	auxiliary function, (46)
$G(f)$	filter gain in decibels, (47)

$\alpha_1(f)$	singular attenuation, (54)
$\alpha_2(f)$	residual attenuation, (54)
$\beta_1(f)$	minimum-phase pair to $\alpha_1(f)$, (56)
$\beta_2(f)$	minimum-phase pair to $\alpha_2(f)$, (56)
K	frequency range of known values, (63)
U	frequency range of unknown values, (63)
s	argument of Laplace transform, (64)
L(s)	Laplace transform of impulse response $h(\tau)$, (64)
$g'(t)$	first derivative of $g(t)$, (A-5)
ω	radian frequency, $2\pi f$, (B-2)

EVALUATION OF ATTENUATION/MINIMUM-PHASE PAIRS BY MEANS OF TWO FAST FOURIER TRANSFORMS

INTRODUCTION

It is often important to determine whether a given linear device is minimum-phase [1], because if so, it is then possible to compensate the filter characteristic with reciprocal pole-zero locations and obtain an overall all-pass characteristic with flat amplitude and linear phase responses. A relatively simple way of making this determination is to measure the attenuation (or decibel gain) and actual phase shift of the given linear device and then compute the minimum-phase corresponding to the measured attenuation. If this latter calculated phase agrees with the actual measured phase, then the filter is minimum-phase.

The minimum-phase corresponding to a given attenuation function is determined analytically by a Hilbert transform [2; chapter 6, article 22] or [3; section 10-3]. However, this direct integral evaluation is computationally unattractive due to two poles on the line of integration [3; (10-67)]. In addition, it yields only a single value for the phase after each numerical integration. We will circumvent both of these difficulties by first subtracting the singularities (which will be handled analytically) and then employing fast Fourier transforms for efficient numerical evaluation of the entire phase response.

TRANSFER FUNCTION RELATIONS

FILTER CHARACTERIZATIONS

A linear time-invariant filter is characterized by its impulse response $h(\tau)$ or by its transfer function $H(f)$ according to Fourier transform

$$H(f) = \int d\tau \, \exp(-i2\pi f\tau) \, h(\tau) \equiv \underline{F}\{h(\tau)\} \, . \tag{1}$$

(Integrals without limits are over the range of nonzero integrand.) Both the impulse response $h(\tau)$ and the transfer function $H(f)$ can be complex functions of time delay τ and frequency f, respectively.

The transfer function will be represented in terms of its real and imaginary parts according to

$$H(f) = H_r(f) + i\, H_i(f) \, , \tag{2}$$

where

$$H_r(f) = \frac{1}{2}[H(f) + H^*(f)] \, ,$$

$$H_i(f) = \frac{1}{i2}[H(f) - H^*(f)] \, . \tag{3}$$

It can also be represented in terms of its even and odd parts as

$$H(f) = H_e(f) + H_o(f) \, , \tag{4}$$

which are generally defined according to

$$H_e(f) = \frac{1}{2}[H(f) + H(-f)] = \int d\tau \cos(2\pi f\tau)\ h(\tau) ,$$

$$H_o(f) = \frac{1}{2}[H(f) - H(-f)] = -i\int d\tau \sin(2\pi f\tau)\ h(\tau) . \qquad (5)$$

Functions $H_e(f)$ and $H_o(f)$ are both complex generally, whereas $H_r(f)$ and $H_i(f)$ are always real. Impulse response $h(\tau)$ can be complex.

(In the special case where impulse response $h(\tau)$ is real, then

$$H_e(f) = H_r(f) = \int d\tau \cos(2\pi f\tau)\ h(\tau) ,$$

$$H_o(f) = i\ H_i(f) = -i\int d\tau \sin(2\pi f\tau)\ h(\tau) .) \qquad (6)$$

CAUSAL FILTER

A filter is said to be causal when its impulse response $h(\tau)$ is zero for negative arguments; that is,

$$h(\tau) = 0 \quad \text{for } \tau < 0 . \qquad (7)$$

However, $h(\tau)$ can still be a complex function of τ. In this causal case, the real and imaginary parts of the transfer function $H(f)$ satisfy a pair of Hilbert transform relationships, provided that $h(\tau)$ does not contain any impulses at the origin; see also [3; page 198]. The Hilbert transform of an arbitrary complex function $G(x)$ is defined as

$$\underline{H}\{G(x)\} = \frac{1}{\pi} ⨍ du \frac{G(u)}{x-u} = \frac{1}{\pi x} \circledast G(x) , \tag{8}$$

where the tic mark on the integral sign denotes a principal value integral [4; section 3.05] and $\circledast$ denotes convolution. Principal value integrals are considered in appendix A.

In order to derive the Hilbert relations of interest, let $U(x)$ be the unit step function,

$$U(x) = \begin{Bmatrix} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{Bmatrix} . \tag{9}$$

Then, because $h(\tau)$ is causal, transfer function (1) becomes

$$H(f) = \int d\tau \exp(-i2\pi f\tau)\, h(\tau)\, U(\tau) = \underline{F}\{h(\tau)\, U(\tau)\} =$$

$$= \underline{F}\{h(\tau)\} \circledast \underline{F}\{U(\tau)\} = H(f) \circledast \left[\frac{1}{2}\delta(f) + \frac{1}{i2\pi f}\right] =$$

$$= \frac{1}{2} H(f) - \frac{i}{2} \underline{H}\{H(f)\} . \tag{10}$$

Here, we used the Fourier transform of the unit step function $U(\tau)$ [3; (3-13)] and definition (8). Equation (10) yields

$$H(f) = -i\, \underline{H}\{H(f)\} \tag{11}$$

or, more explicitly,

$$H_r(f) = \underline{H}\{H_i(f)\} = \frac{1}{\pi f} \circledast H_i(f) ,$$

$$H_i(f) = - \underline{H}\{H_r(f)\} = - \frac{1}{\pi f} \circledast H_r(f) . \tag{12}$$

We repeat that transfer function relations (12) hold true even when impulse response $h(\tau)$ is complex; only causality is used. Analogous properties to (12) hold between the even and odd parts, $H_e(f)$ and $H_o(f)$, of the transfer function $H(f)$ as well. Namely, because the Hilbert transform of an even (odd) function is odd (even), there follows, for a causal (but possibly complex) $h(\tau)$,

$$H_e(f) = -i\ \underline{H}\{H_o(f)\}\ , \qquad H_o(f) = -i\ \underline{H}\{H_e(f)\}\ . \tag{13}$$

If $h(\tau)$ contains an impulse at the origin, both parts of (12) are false, even though $h(\tau)$ may be causal. Consider

$$h(\tau) = (a + ib)\ \delta(\tau), \qquad a \text{ and } b \text{ real}\ . \tag{14}$$

Then (1) yields constant transfer function

$$H(f) = a+ib,\ H_r(f) = a,\ H_i(f) = b,\ H_e(f) = a+ib,\ H_o(f) = 0. \tag{15}$$

But since the Hilbert transform of a constant is zero [4; section 3.05], neither part of (12) is satisfied, and the first part of (13) is false. On the other hand, if

$$h(\tau) = (a + ib)\ \delta(\tau - T)\ , \qquad a \text{ and } b \text{ real}\ , \tag{16}$$

then (12) and (13) are satisfied only if $T > 0$. Here, we used the facts that

$$\underline{H}\{\cos(2\pi fT)\} = \sin(2\pi f|T|),\ \underline{H}\{\sin(2\pi fT)\} = -\mathrm{sgn}(T)\ \cos(2\pi fT), \tag{17}$$

where $\mathrm{sgn}(T)$ is the polarity of T. Henceforth, we assume that components like (14) and (15) are not present in the filters of interest; see also [3; page 198].

For a causal filter, (2) and (12) afford a method of obtaining the complete transfer function from its real part alone, according to

$$H(f) = H_r(f) + i\, H_i(f) =$$

$$= H_r(f) - i\, \underline{H}\{H_r(f)\}\ . \tag{18}$$

However, a more attractive approach, computationally, is to use Fourier transforms, as follows. Define inverse Fourier transform

$$\underline{h}(\tau) \equiv \underline{F}^{-1}\{H_r(f)\} = \int df\, \exp(i2\pi f\tau)\, H_r(f) \tag{19}$$

for any real part $H_r(f)$. (The notation $h_r(\tau)$ cannot be used instead of $\underline{h}(\tau)$, because $\underline{h}(\tau)$ is not the real part of $h(\tau)$, nor is $\underline{h}(\tau)$ necessarily real.) Substitution of (3) into (19) immediately yields

$$\underline{h}(\tau) = \frac{1}{2}\left[h(\tau) + h^*(-\tau)\right]\ ; \quad \underline{h}(-\tau) = \underline{h}^*(\tau)\ . \tag{20}$$

(These particular relations in (20) actually hold true for any filter $h(\tau)$, noncausal as well as complex.) Then because $h(\tau)$ is causal, there follows directly

$$h(\tau) = \left\{\begin{matrix} 2\underline{h}(\tau) & \text{for } \tau > 0 \\ 0 & \text{for } \tau < 0 \end{matrix}\right\} = 2\, \underline{h}(\tau)\, U(\tau)\ . \tag{21}$$

In summary, the method for obtaining the complete transfer function $H(f)$ from just its real part $H_r(f)$, for a causal filter, is to perform, in order, the following operations:

$$\underline{h}(\tau) = \underline{F}^{-1}\{H_r(f)\} ,$$
$$h(\tau) = 2\ \underline{h}(\tau)\ U(\tau) ,$$
$$H(f) = \underline{F}\{h(\tau)\} . \qquad (22)$$

This procedure requires two Fourier transforms, which can be accomplished very quickly and efficiently by means of two fast Fourier transforms. Furthermore, a fast Fourier transform output sweeps out the complete range of argument values, whereas the brute force Hilbert transform integral of (18) and (8) requires an additional numerical integration for each frequency f of interest. Functions $h(\tau)$ and $\underline{h}(\tau)$ in (22) can be complex.

An accuracy check on the procedure in (22) is afforded by comparing the real part output of the Fourier transform in the bottom line with the input $H_r(f)$ utilized in the top line. The complete set of function values of $H_r(f)$ for all f is required for this procedure; in return, the complete set of values of $H_i(f)$, for all f, results. The operations in (22) are linear insofar as the overall transformation of $H_r(f)$ is concerned, and so superposition can be used for any breakdown of $H_r(f)$ into components, if desired.

The rule for obtaining $H(f)$ or $H_i(f)$ from $H_r(f)$, as given in (22), applies whether filter $H(f)$ is minimum-phase [1] or not. The only prerequisite for the validity of (22) is the causality of impulse response $h(\tau)$.

If only $H_e(f)$ were available (instead of $H_r(f)$), a more attractive procedure for obtaining $H(f)$ or $H_i(f)$ than using (4) and Hilbert transform (13), is to observe that, in general, for <u>any</u> filter, the inverse Fourier transform

$$\underline{F}^{-1}\{H_e(f)\} = \int df \exp(i2\pi f\tau)\, H_e(f) = \tfrac{1}{2}[h(\tau)+h(-\tau)] = h_e(\tau). \quad (23)$$

Here, we used (5), the inverse to (1), and the general definition of the even part of an arbitrary complex function. Then, if $h(\tau)$ is causal, we have

$$h(\tau) = 2\, h_e(\tau)\, U(\tau)\ . \quad (24)$$

Thus, the procedure for obtaining $H(f)$ is identical to (22) if we replace $H_r(f)$ and $\underline{h}(\tau)$ by $H_e(f)$ and $h_e(\tau)$, respectively.

ONE-SIDED SPECTRAL FUNCTIONS

The analogous situation in the frequency domain (to causality in the time delay domain) is as follows: if (complex function) $A(f)$ is zero for negative arguments, that is,

$$A(f) = 0 \quad \text{for } f < 0\ , \quad (25)$$

then a procedure similar to (10)-(11) reveals that the inverse Fourier transform of $A(f)$ is given by

$$a(\tau) \equiv \underline{F}^{-1}\{A(f)\} = i\, \underline{H}\{a(\tau)\}\ . \quad (26)$$

That is, in terms of real and imaginary parts,

$$a_r(\tau) = -\,\underline{H}\{a_i(\tau)\}\ , \quad a_i(\tau) = \underline{H}\{a_r(\tau)\}\ . \quad (27)$$

The function $a(\tau)$ is called an analytic waveform, for reasons to become apparent shortly.

GENERAL SPECTRAL RELATIONS

For future purposes, the Hilbert transform of a completely arbitrary complex waveform $b(\tau)$,

$$b_H(\tau) \equiv \underline{H}\{b(\tau)\} = \frac{1}{\pi\tau} \circledast b(\tau) , \qquad (28)$$

has spectrum (Fourier transform)

$$\underline{F}\{b_H(\tau)\} = -i \; \mathrm{sgn}(f) \; B(f) = \begin{Bmatrix} -i \; B(f) \text{ for } f > 0 \\ i \; B(f) \text{ for } f < 0 \end{Bmatrix} , \qquad (29)$$

where $B(f)$ is the spectrum of $b(\tau)$. Here, we used the fact that the following two functions are a Fourier transform pair [3; apply (2-34) to (3-9)]:

$$\frac{1}{\pi\tau} \longleftrightarrow -i \; \mathrm{sgn}(f) . \qquad (30)$$

The left-hand side of (29) is the Fourier transform of the Hilbert transform of $b(\tau)$. It cannot be labeled as $B_H(f)$, which is the Hilbert transform of the Fourier transform $B(f)$ of $b(\tau)$. The two operations of Hilbert transformation and Fourier transformation are not interchangeable, in general.

It follows from (29) that

$$\underline{F}\{b(\tau) + i \; b_H(\tau)\} = 2 \; B(f) \; U(f) , \qquad (31)$$

which is a one-sided spectrum. Also, $b(\tau) + i \; b_H(\tau)$ is an analytic waveform. Waveform $b(\tau)$ is completely arbitrary here.

ANALYTICITY OF TRANSFER FUNCTION

Consider the causal exponential impulse response

$$h(\tau) = \exp(-\tau)\ U(\tau)\ . \tag{32}$$

The corresponding transfer function is

$$H(f) = \frac{1}{1 + i2\pi f}\ , \tag{33}$$

which has a pole in the upper-half f-plane at $f = i/(2\pi)$, but which is analytic in the lower-half f-plane. (The lower-half f-plane corresponds to the right-half s-plane of Laplace transforms.)

This analyticity of the transfer function H(f) in the lower-half f-plane is generally true for causal finite-energy filters, as may be seen by the following argument. Let frequency f be a complex variable with real and imaginary parts according to $f = f_r + if_i$. Then, for a causal filter, (1) can be expressed more explicitly as

$$H(f) = \int_0^{+\infty} d\tau\ \exp(-i2\pi f_r \tau)\ \exp(2\pi f_i \tau)\ h(\tau)\ . \tag{34}$$

The first exponential in (34) has magnitude 1 for all τ on the contour of integration. And if $f_i < 0$, the second exponential term in (34) decays with increasing τ, keeping the integral convergent, as it was for $f_i = 0$. That is, transfer function H(f) is analytic in the lower-half f-plane for a causal impulse response $h(\tau)$. Notice, however, that no statements can be made about the locations of the zeros of transfer function H(f) in the complex f-plane. Thus we have

$$\text{causal } h(\tau) \longrightarrow \text{analytic } H(f) \text{ in lower-half f-plane} . \qquad (35)$$

The converse is also true, namely, that analyticity implies causality. To develop this point, express the inverse Fourier transform to (1) in the form

$$h(\tau) = \int_{C_1} df \, \exp(i2\pi f\tau) \, H(f) =$$

$$= \int_{C_2} df \, \exp(i2\pi f_r\tau) \, \exp(-2\pi f_i\tau) \, H(f) , \qquad (36)$$

where contours C_1 and C_2 are depicted in the complex f-plane in figure 1. Because transfer function H(f) is analytic in the (crosshatched) region between contours C_1 and C_2, we are allowed to move the integration freely between them, as done in (36),

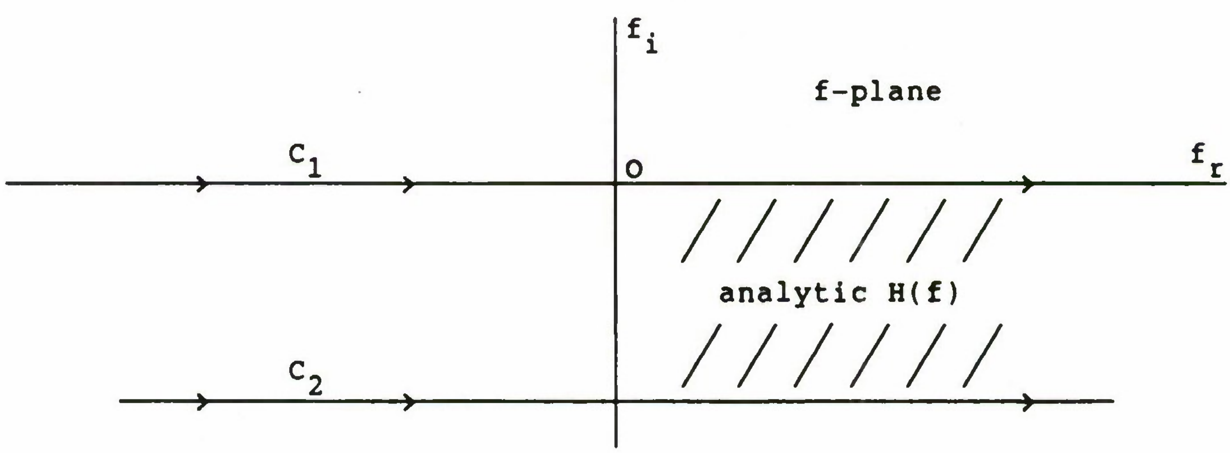

Figure 1. Complex f-Plane Contours

without altering the value $h(\tau)$ of the integral. On contour C_2, we have $f_i < 0$ everywhere. Therefore, if $\tau < 0$ in (36), the second exponential decays to zero as contour C_2 is moved farther down in the f-plane. Because $H(f)$ is analytic in the lower-half f-plane, we can move C_2 arbitrarily far down, causing the integrand of (36) to go to zero, thereby leading to a zero value for $h(\tau)$ whenever $\tau < 0$. Thus, we have

$$\text{analytic } H(f) \text{ in lower-half f-plane} \longrightarrow \text{causal } h(\tau) \; . \tag{37}$$

This equation is the converse to (35).

Because we have already shown in (10)-(12) that a causal impulse response $h(\tau)$ leads to a transfer function $H(f)$ with Hilbert transform relations between its real and imaginary parts, it follows from (37) that an analytic transfer function $H(f)$ leads to the same conclusions. This means that, for an <u>analytic</u> transfer function $H(f)$ in the lower-half f-plane, we can use the efficient procedure given in (22), in terms of two (fast) Fourier transforms, to find the imaginary part $H_i(f)$, given only the real part $H_r(f)$.

For the example given earlier in (33), we have real part

$$H_r(f) = \frac{1}{1 + (2\pi f)^2} \; .$$

Then from (22), we obtain, in order,

$$\underline{h}(\tau) = \frac{1}{2}\exp(-|\tau|) \; , \quad h(\tau) = \exp(-\tau)\, U(\tau) \; , \quad H(f) = \frac{1}{1 + i2\pi f} \; ,$$

which corroborates (32) and (33).

MINIMUM-PHASE TRANSFER FUNCTIONS

From this point on, we presume that impulse response $h(\tau)$ is causal and that transfer function $H(f)$ contains only poles and zeros. It then follows from (35) that transfer function $H(f)$ has no poles in the lower-half f-plane. We also assume now that $H(f)$ has no zeros in the lower-half f-plane; that is, the filter is minimum-phase [1,2,3]. In this case, the function

$$Q(f) = -\ln H(f) \tag{38}$$

is analytic in the lower-half f-plane, because the function $\ln z$ is nonanalytic only at $z = 0$ and $z = \infty$ in the complex z-plane. Accordingly, by analogy to (37), inverse Fourier transform

$$q(\tau) = \int df \exp(i2\pi f\tau)\, Q(f) \tag{39}$$

is causal. (An example is given in appendix B.) Therefore, just as shown in (10)-(12), the real and imaginary parts of $Q(f)$,

$$Q(f) = Q_r(f) + i\, Q_i(f) \ , \tag{40}$$

can be found from each other by means of Hilbert transforms. In particular, as in (12),

$$Q_r(f) = \underline{H}\{Q_i(f)\} \ , \quad Q_i(f) = -\underline{H}\{Q_r(f)\} \ . \tag{41}$$

Alternatively, according to the sequel to (37), because $Q(f)$ is analytic in the lower-half f-plane, the imaginary part $Q_i(f)$ can be found from real part $Q_r(f)$ according to procedure (22) involving two Fourier transforms.

Interesting interpretations of minimum-phase filters, in terms of their group delay and rate of energy flow through the filter, are given in [5; pages 132-3]. In particular, the minimum-phase filter has the smallest group delay of any stable filter with specified magnitude transfer function.

ATTENUATION AND PHASE

There is another way of describing a transfer function H(f) rather than by its real and imaginary parts, which is very useful in some applications. Namely, let

$$H(f) = \exp[-\alpha(f) - i\,\beta(f)] \;, \tag{42}$$

where

$$\left.\begin{aligned} \alpha(f) &= \text{attenuation} \\ \beta(f) &= \text{phase shift} \end{aligned}\right\} \text{ of filter} \;. \tag{43}$$

Reference to (38) and (40) immediately reveals that

$$\alpha(f) = Q_r(f) \;, \quad \beta(f) = Q_i(f) \;. \tag{44}$$

Therefore, if filter H(f) is minimum-phase, according to the discussion in (38)-(41), $\alpha(f)$ and $\beta(f)$ can be found from each other by means of Hilbert transforms. In particular,

$$\beta(f) = -\,\underline{H}\{\alpha(f)\} = -\,\frac{1}{\pi f} \circledast \alpha(f) \;. \tag{45}$$

(Strictly, this relation is not usable and must be modified to allow for attenuations $\alpha(f)$ with logarithmic singularities; for example, see [3; pages 206-8]. This manipulation is discussed in appendix C.)

Alternatively, the procedure in (22) can be employed in the form

$$\underline{g}(\tau) = \underline{F}^{-1}\{\alpha(f)\} ,$$
$$q(\tau) = 2\, \underline{g}(\tau)\, U(\tau) ,$$
$$\alpha(f) + i\, \beta(f) = \underline{F}\{q(\tau)\} . \tag{46}$$

The function $\underline{g}(\tau)$ is defined by the inverse Fourier transform in the top line of (46). Phase shift $\beta(f)$ for a minimum-phase filter is given by the imaginary part of the Fourier transform in the bottom line of (46).

A common alternative descriptor of the frequency behavior of a filter is the gain $G(f)$ in decibels, defined as

$$G(f) = 20 \log_{10} |H(f)| . \tag{47}$$

Because the attenuation follows from (42) as

$$\alpha(f) = - \ln |H(f)| , \tag{48}$$

the gain $G(f)$ and the attenuation $\alpha(f)$ are related by

$$G(f) = - \frac{20}{\ln(10)}\, \alpha(f) = - 8.686\, \alpha(f) . \tag{49}$$

Measurement of either one is sufficient to find the other and to thereby determine the phase shift $\beta(f)$ of a minimum-phase filter.

EXAMPLE AND LIMITATION

We again consider the example given in (32)-(33), namely

$$h(\tau) = \exp(-\tau)\, U(\tau) \ , \quad H(f) = \frac{1}{1 + i2\pi f} \ . \tag{50}$$

The attenuation and phase follow from (42) according to

$$\alpha(f) = \frac{1}{2} \ln(1 + 4\pi^2 f^2) \ ,$$

$$\beta(f) = \arctan(2\pi f) \ . \tag{51}$$

If we attempt to apply the inverse Fourier transform in the top line of (46) to the attenuation $\alpha(f)$ in (51), we encounter a divergent integral because $\alpha(f) \sim \ln|f|$ as $f \to \pm\infty$.

More generally, if filter $H(f)$ has a zero at a frequency f equal to any finite real value, the attenuation $\alpha(f)$ has a logarithmic singularity at that real frequency, and the inverse Fourier transform in (46) diverges. Because typical filters very often have this feature (and almost always at $f = 0$ and $f = \pm\infty$), a way must be found to circumvent the divergent part of the inverse Fourier transform integral, so that the efficient procedure of (46) can be salvaged.

SUBTRACTION OF SINGULARITY

The procedure to be used here is one commonly adopted to numerically evaluate convergent integrals with singular integrands; it is illustrated by the example

$$I = \int_0^a dx \, \frac{\cos x}{x^\nu} \, , \qquad \nu < 1 \, . \tag{52}$$

If ν is positive, the integrand has an infinite cusp at the origin, yet the integral converges, because $\nu < 1$. We express

$$I = \int_0^a dx \, \frac{\cos x - 1 + 1}{x^\nu} = \int_0^a dx \, \frac{\cos x - 1}{x^\nu} + \int_0^a dx \, \frac{1}{x^\nu} \, , \tag{53}$$

which is allowed, because both integrals converge. The last integral in (53) can be done in closed form, yielding $a^{1-\nu}/(1-\nu)$. Also, the middle integrand now behaves as $x^{2-\nu}$ as $x \to 0+$, which is zero at the origin, because $2-\nu > 1$; this behavior enables a straightforward numerical evaluation of the middle integral.

The key to this procedure is to find a component that can be integrated in closed form and that, when subtracted from the given integrand, yields a well-behaved residual for numerical integration.

APPLICATION TO FILTERS

The way we apply this subtraction procedure to a given attenuation $\alpha(f)$ with logarithmic singularities is to break it into two parts,

$$\alpha(f) = \alpha_1(f) + \alpha_2(f) \ , \tag{54}$$

where attenuation $\alpha_1(f)$ contains all the singular components and has a known closed form minimum-phase pair $\beta_1(f)$. (An example is furnished by (50) and (51); some additional examples are listed in appendix D.) Then residual attenuation $\alpha_2(f)$ is found according to

$$\alpha_2(f) = \alpha(f) - \alpha_1(f) \tag{55}$$

and is well-behaved for all f. Residual $\alpha_2(f)$ is subjected to the repeated Fourier transform procedure detailed in (46), resulting in phase shift function $\beta_2(f)$. Finally, the complete minimum-phase corresponding to the given attenuation $\alpha(f)$ is obtained from

$$\beta(f) = \beta_1(f) + \beta_2(f) \ . \tag{56}$$

The procedure can be summarized as follows:

$$\alpha(f) \longrightarrow \beta(f) \quad \text{desired} \ ;$$

$$\alpha_1(f) + \alpha_2(f) \longrightarrow \beta_1(f) + \beta_2(f) \quad \text{used} \ . \tag{57}$$

The exact choice of attenuation/minimum-phase pair $\alpha_1(f)$, $\beta_1(f)$ is not critical, except that residual $\alpha_2(f)$ must not have any singularities and must decay (rapidly) to zero for large f.

Of course, the given attenuation $\alpha(f)$ must be known for all f in order to apply this (or any) procedure for obtaining minimum-phase shift $\beta(f)$, whether obtained directly by Hilbert transforms or by means of a Fourier procedure. The actual numerical evaluation of the Fourier procedure delineated in (46) is accomplished by means of fast Fourier transforms; the details are presented in appendix E.

SHORTCOMING OF HILBERT TRANSFORM

Suppose that two minimum-phase filters $H_a(f)$ and $H_b(f)$ differ only by a complex scale factor:

$$H_b(f) = c\, H_a(f)\ . \tag{58}$$

Then

$$\alpha_b(f) = \alpha_a(f) - \ln|c|\ ,$$

$$\beta_b(f) = \beta_a(f) - \arg(c) + 2\pi n\ , \quad n \text{ integer}\ . \tag{59}$$

However, if $\alpha_a(f)$ and $\beta_a(f)$ are a Hilbert transform pair, $\alpha_b(f)$ and $\beta_b(f)$ cannot possibly be (unless $c = 1$ and $n = 0$) because the Hilbert transform of a constant is zero. Functions $\alpha_b(f)$ and $\beta_b(f)$ are both "incomplete," in that attenuation $\alpha_b(f)$ contains no information about arg(c), while phase $\beta_b(f)$ contains no information about $|c|$. This means that the Hilbert transform of a given attenuation (phase) yields a phase (attenuation) function that can differ from the actual phase (attenuation) of a minimum-phase filter by an arbitrary additive constant. Some information is inherently absent from a given attenuation (phase) function. In addition, because the Hilbert transform of a constant is zero, additive constants are lost through this transformation. (The situation is somewhat similar for the Fourier transform procedure given in (46).)

Alternatively, suppose that

$$h_b(\tau) = h_a(\tau - T) \ , \quad H_b(f) = H_a(f) \exp(-i2\pi fT) \ . \tag{60}$$

Then filter $H_b(f)$ contains a transfer function component of $\exp(-i2\pi fT)$, with corresponding attenuation 0 and phase $2\pi fT$. Thus, the attenuation contains no information about a pure time delay. However, it should be noted that this component $\exp(-i2\pi fT)$ does not possess poles and zeros at all, but in fact has an essential singularity at $f = \infty$.

APPLICATION TO MEASURED DATA

In this section, we will apply the previous Fourier procedure to a measured pair of attenuation and phase shift functions in an effort to determine if the filter is minimum-phase. The particular filter is a J15-1 transducer used as a continuous-wave source in the 10 to 900 Hertz range. The transmitting current response of this device is defined as the ratio

$$\frac{\text{output pressure}}{\text{input current}} \tag{61}$$

and is the transfer function of interest. The reference level is taken as 1 μPa/Amp. The measurements procedure include a water-path propagation delay (of unknown value) between the transducer and a calibrated receiving hydrophone.

The measured decibel gain, (47)-(49), of transfer function (61) is displayed in figure* 2 for the range of frequencies from 30 to 500 Hertz, on a logarithmic frequency abscissa. Also superposed are the decibel gain responses of filters with 1 or 2 or 3 poles at the origin, which plot as straight lines on this type of paper. This information is required for determining the behavior of the filter from 30 Hertz down to f = 0 and is necessary because the Hilbert and Fourier procedures both require knowledge of the complete attenuation (or gain) for <u>all</u> frequencies, in order to determine the value of the corresponding minimum-phase shift at just one frequency. It may be reasonably concluded from the fits in figure 2 that the transducer of

*Figures 2 through 11 are collected at the end of this section.

interest here has a double zero at $f = 0$.

In addition, the same fitting procedure has been attempted in the neighborhood of 500 Hertz in figure 2, as may be seen by the superposition of responses for filters with decays corresponding to 0 or 1 or 2 or 3 poles at $f = \infty$. However, the situation is rather poor at this upper end of the measured frequency range, because, as seen in figure 2, the transducer has not yet developed its asymptotic behavior at $f = 500$ Hertz. This behavior is consistent with the information mentioned above, which describes the use of this device as a source up to 900 Hertz. Thus, we have a situation where we have insufficient measurements to fully apply the theoretical developments presented earlier. Nevertheless, we will attempt to circumvent the inadequacy by extrapolating the given measurements into the frequency range above 500 Hertz and then using the combination of measured and extrapolated gains to determine the minimum-phase response.

PHILOSOPHY OF EXTRAPOLATION

A situation of frequent occurrence is the following. We have a measured residual attenuation $\alpha_2(f)$, but it is available only for $0 \le f_1 < f < f_2$; see (54)-(57). We presume that attenuation $\alpha_2(f)$ is even about $f = 0$. Call this total frequency range of known values, K. Denote the remainder of the frequency range, where $\alpha_2(f)$ is unknown, by U.

We want to evaluate the minimum-phase corresponding to $\alpha_2(f)$, namely

$$\beta_2(f) = -\underline{H}\{\alpha_2(f)\} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} du\, \frac{\alpha_2(u)}{f - u} \,. \qquad (62)$$

Our approach is to extrapolate $\alpha_2(f)$ beyond K into the unknown frequency range U. Call this extrapolated function $\alpha_{2e}(f)$; it exists for all f. This extrapolation must be rather close to the true (unknown) attenuation $\alpha_2(f)$ in U, but $\alpha_{2e}(f)$ need not agree with $\alpha_2(f)$ inside K. In particular, $\alpha_{2e}(f)$ and $\alpha_2(f)$ should match in value and slope at the boundaries of K.

Then, we can obtain the following approximation to phase (62), namely

$$\beta_{2a}(f) \equiv -\frac{1}{\pi}\int_K du\, \frac{\alpha_2(u)}{f - u} - \frac{1}{\pi}\int_U du\, \frac{\alpha_{2e}(u)}{f - u} =$$

$$= -\frac{1}{\pi}\int_K du\, \frac{\alpha_2(u) - \alpha_{2e}(u)}{f - u} - \frac{1}{\pi}\int_{-\infty}^{+\infty} du\, \frac{\alpha_{2e}(u)}{f - u} \,. \qquad (63)$$

The first (finite) integral in (63) is done numerically, by employing the Fourier procedure presented here. The second integral in (63) is actually divergent and is instead replaced by use of a known attenuation/minimum-phase pair, $\alpha_{2e}(f)$, $\beta_{2e}(f)$.

The key to this procedure is a shrewd choice for the extrapolated attenuation $\alpha_{2e}(f)$. Several candidates, along with the corresponding minimum-phase functions, are listed in appendix D.

LAPLACE TRANSFORM NOTATION

For convenience of notation, we employ here the Laplace transform of the impulse response, namely

$$L(s) = \int_0^{+\infty} d\tau \exp(-s\tau)\ h(\tau) \ , \tag{64}$$

where we have specifically limited consideration to causal filters. The connection with the Fourier transform (1) is

$$H(f) = L(i2\pi f) \ . \tag{65}$$

EXAMPLE A

The first attempted fit to the measured gain in figure 2 is by means of filter

$$L(s) = \frac{c\ s^2}{(s + a)(s + b)} \ , \tag{66}$$

with constants a = 260, b = 330, and c = - .55E8. This filter has the desired double-order zero at the origin, but does not decay for large frequencies. The gain of (66) is superposed on the measured gain in figure 3; it is seen that the constants have been chosen to give a fit that matches in value and slope for small frequencies and that matches the measured gain value at 500 Hertz.

The difference in decibels between the measured gain and the fitted gain is displayed in figure 4; it goes to zero at 30 and 500 Hertz and is assumed to be zero outside this range. This assumption is not likely to be correct for f greater than 500 Hertz, but it is necessary in order to proceed with the numerical manipulations. The difference in attenuations, $\alpha_2(f)$ of (55), is available by dividing the result in figure 4 by -8.686; see (47)-(49).

The residual attenuation $\alpha_2(f)$ is subjected to the cascaded Fourier procedure of (46), and the resultant phase $\beta_2(f)$ is added to the minimum-phase $\beta_1(f)$ corresponding to (66). The final total phase $\beta(f)$ is shown in figure 5, with the label A&T, meaning analytic and transform, that is, $\beta_1(f)$ plus $\beta_2(f)$. Superposed on this figure is the measured phase, with the label M&D, meaning measured and time-delay adjusted. Recall in the discussion surrounding (61) that there is an unknown time delay, between the transducer and receiving hydrophone, included in the measurements taken. Accordingly, a selection of time delay was made that yielded the best eyeball fit of the two phases over the range of frequencies from 0 to 400 Hertz in figure 5; this corresponds to an additive linear phase function of frequency, as indicated by example (60). The time delay was 1.43 ms.

The agreement between the minimum-phase and the measured results in figure 5 allow us to conclude that the J15-1 transducer is indeed a minimum-phase filter, at least over the frequency range up to 400 Hertz. The difference between the two results is 17° at 500 Hertz, which is significant. However, the reason for this discrepancy is undoubtedly due to the fact that (66) is not the correct fit for f > 500 Hertz, because (66) has no decay for large frequencies.

EXAMPLE B

In an effort to find a better phase match, another fit was also tried, namely filter

$$L(s) = \frac{c\, s^2}{(s + a_o)[(s + a)^2 + b^2]} , \qquad (67)$$

with constants $a_o = 4000$, $a = 260$, $b = 400$, and $c = -.275E12$. The measured and analytical decibel gains are plotted in figure 6, while the decibel difference is plotted in figure 7. The corresponding two phase plots, obtained by an identical procedure to that described in example A above, are presented in figure 8. Now, the difference in the two phase curves at 500 Hertz has decreased, but only slightly, to 14°. Apparently, the unmeasured decibel gain, in the frequency range above 500 Hertz, is causing inaccurate calculations of the minimum-phase in the region just below 500 Hertz, due to our inability to correctly extrapolate, by means of (66) and (67), to what the filter gain truly was in that frequency range. This supposition is consistent with the observation that the minimum-phase at a particular frequency is largely governed by the (rate of change of the) attenuation in the neighborhood of that frequency [2; page 345]. The agreement in phase results for the lower frequencies comes about because errors in gain measurements above 500 Hertz have a much reduced effect on the calculated phase at low frequencies.

EXAMPLE C

In an attempt to justify this conjecture, an estimate of the unmeasured gain in the frequency range from 500 to 900 Hertz was made and is illustrated in figure 9. A droop of 7 dB, centered at 565 Hertz, has been added and is annotated by the phrase "augmented". The fit is again (66), with the same constants as used for example A, and is superposed in the figure.

The two phase curves are illustrated in figure 10. Now, the discrepancy between the two results is negligible (within measurement error) all the way up to 500 Hertz, the maximum frequency at which the phase was measured. Thus, we feel justified in concluding that the device under investigation is indeed a minimum-phase filter, at least over the measured frequency range up to 500 Hertz.

LIMITED FREQUENCY RANGE

It has been stated above that the measured filter appears to be minimum-phase in a particular frequency range. Strictly, this is not a valid concept; but it is necessary to allow for it in practice, where filter responses cannot possibly be measured for all frequencies. For example, suppose that the transfer function H(f) has a collection of poles and zeros in the upper-half f-plane, all fairly near the origin $f = 0$. In addition, let H(f) have a pole-zero pair far away from the origin, but symmetrically located about the real f axis, so as not to affect the gain or attenuation; see the pair near $f = f_2$ in figure 11.

Obviously, the filter in figure 11 cannot be minimum-phase, because it has a zero in the lower-half f-plane. Yet, its measured phase, for frequencies less than f_1, would be indistinguishable from that of the minimum-phase filter that does not contain that extra pair. Thus, we would reasonably conclude, upon the basis of the measurements made, that the filter is "minimum-phase for $f < f_1$." Furthermore, this is a practically useful concept because compensation of the filter in this same frequency range is certainly possible and allowable. In other words, measurement in a limited frequency range only allows us to make conclusions in that same range; in fact, the situation is slightly worse than that, because the edges of the range may also be open to question.

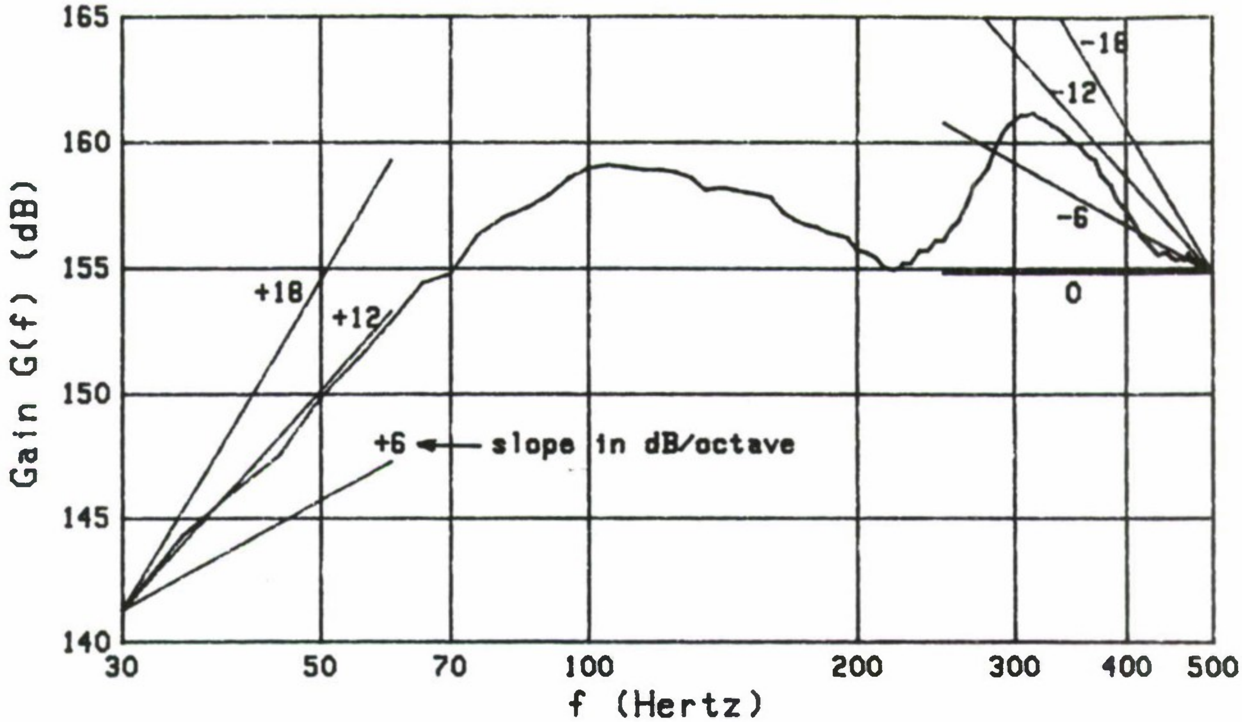

Figure 2. Measured Filter Gain

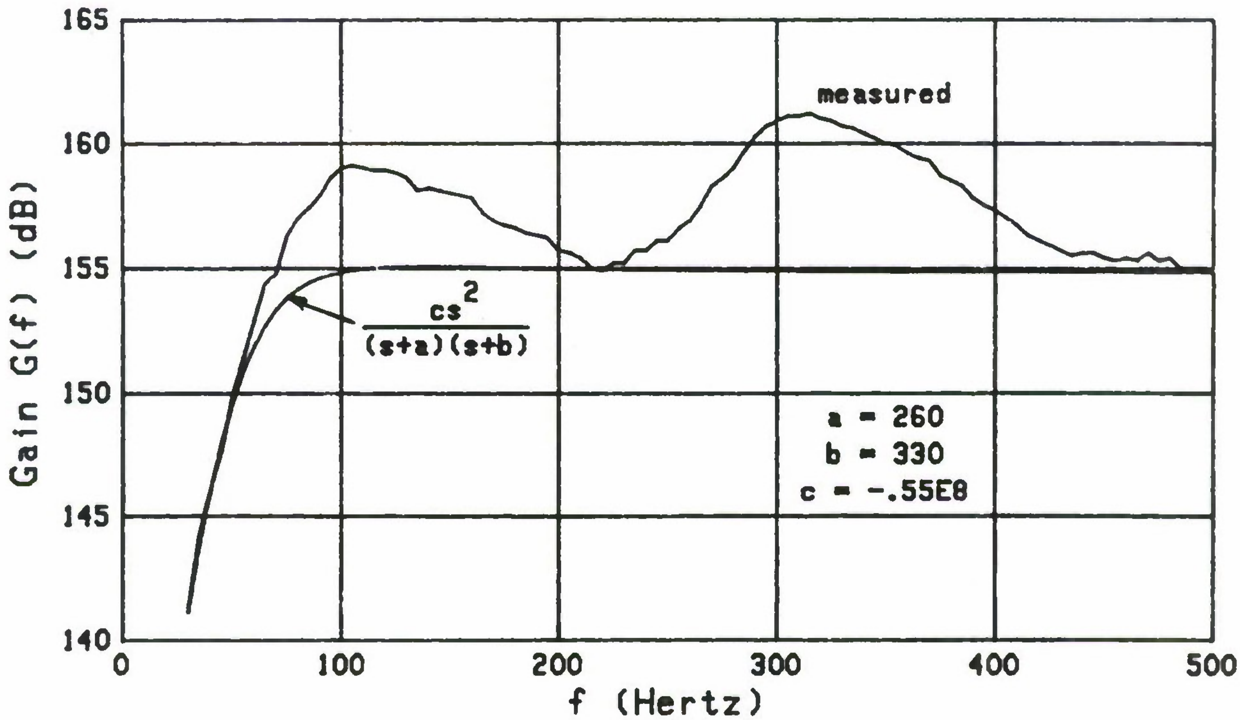

Figure 3. Fitted Gain for Example A

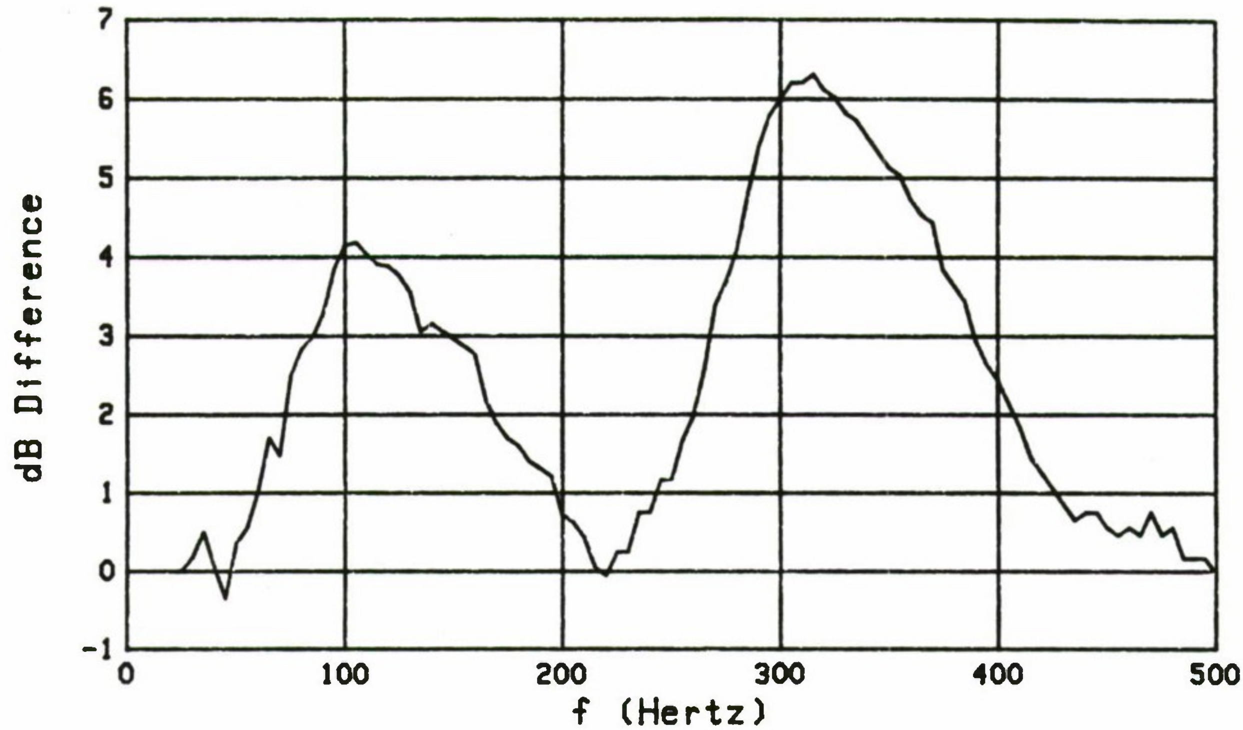

Figure 4. Decibel Difference for Example A

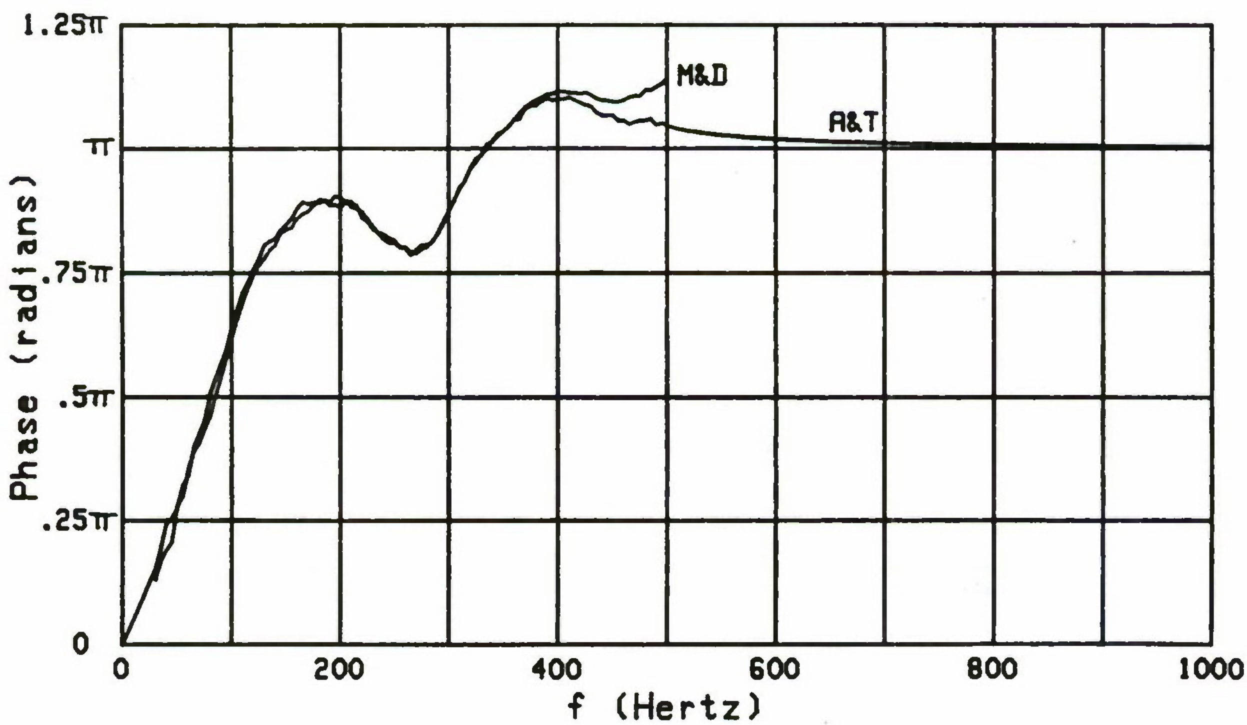

Figure 5. Measured and Transformed Phases for Example A

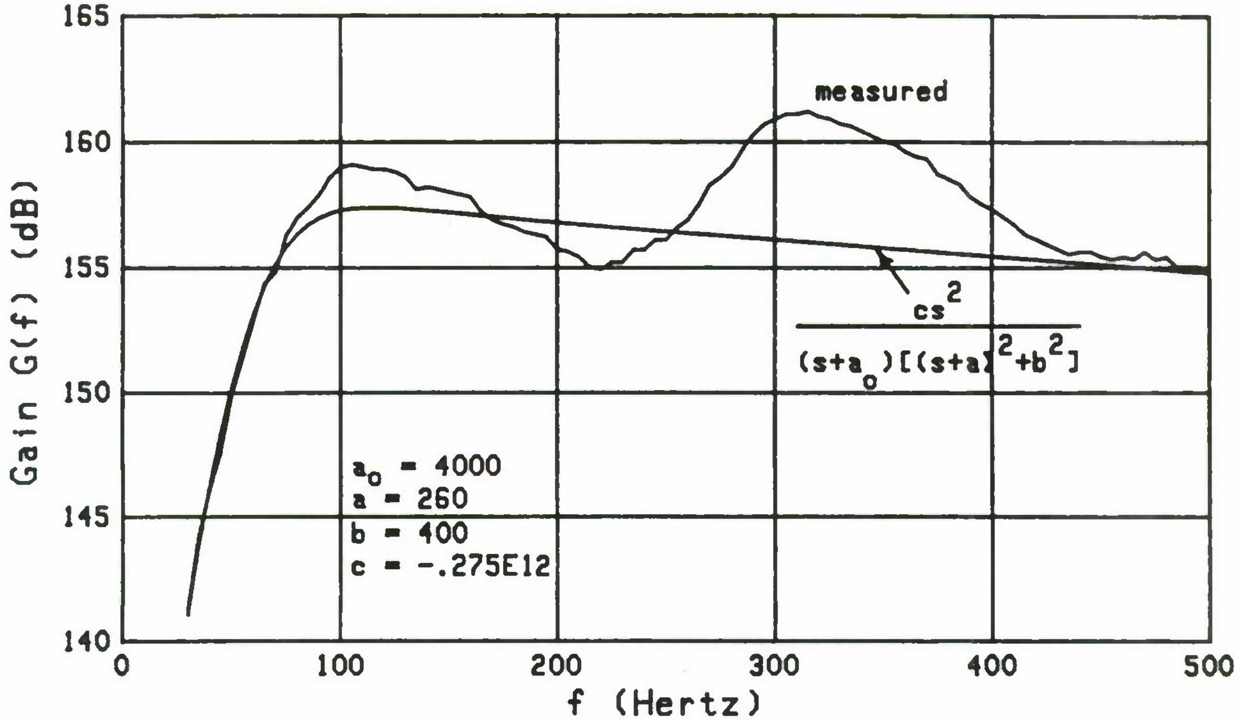

Figure 6. Fitted Gain for Example B

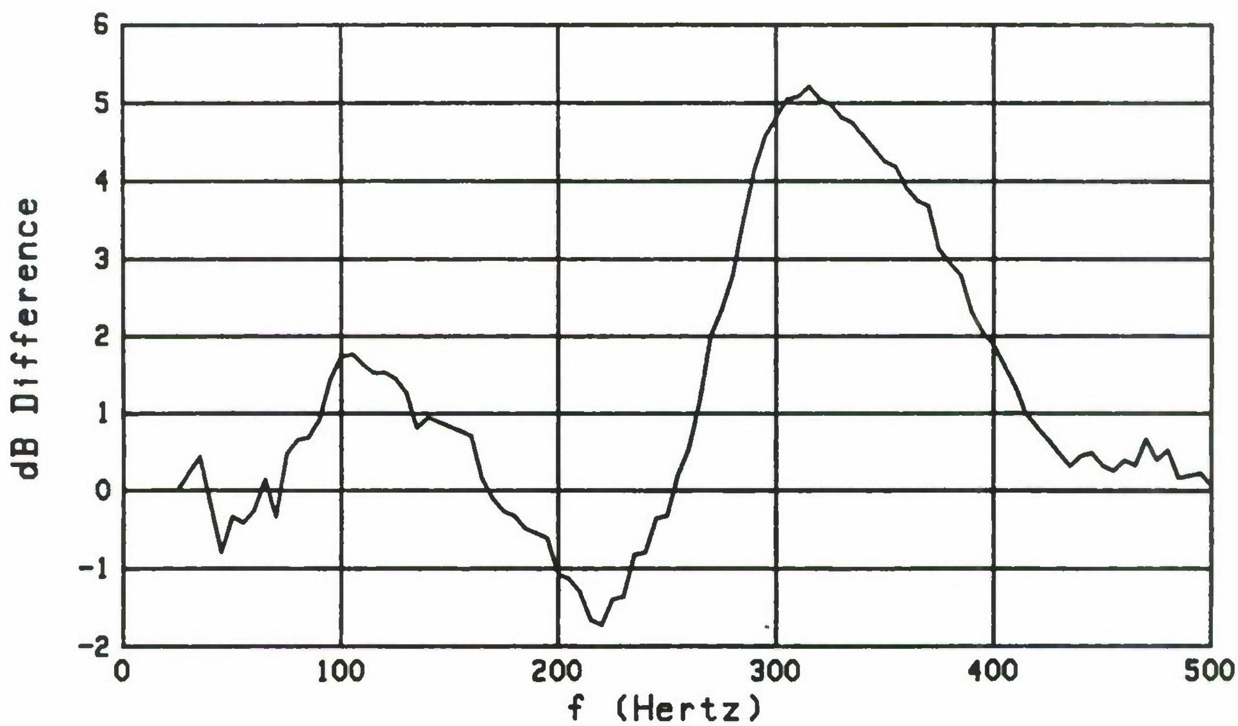

Figure 7. Decibel Difference for Example B

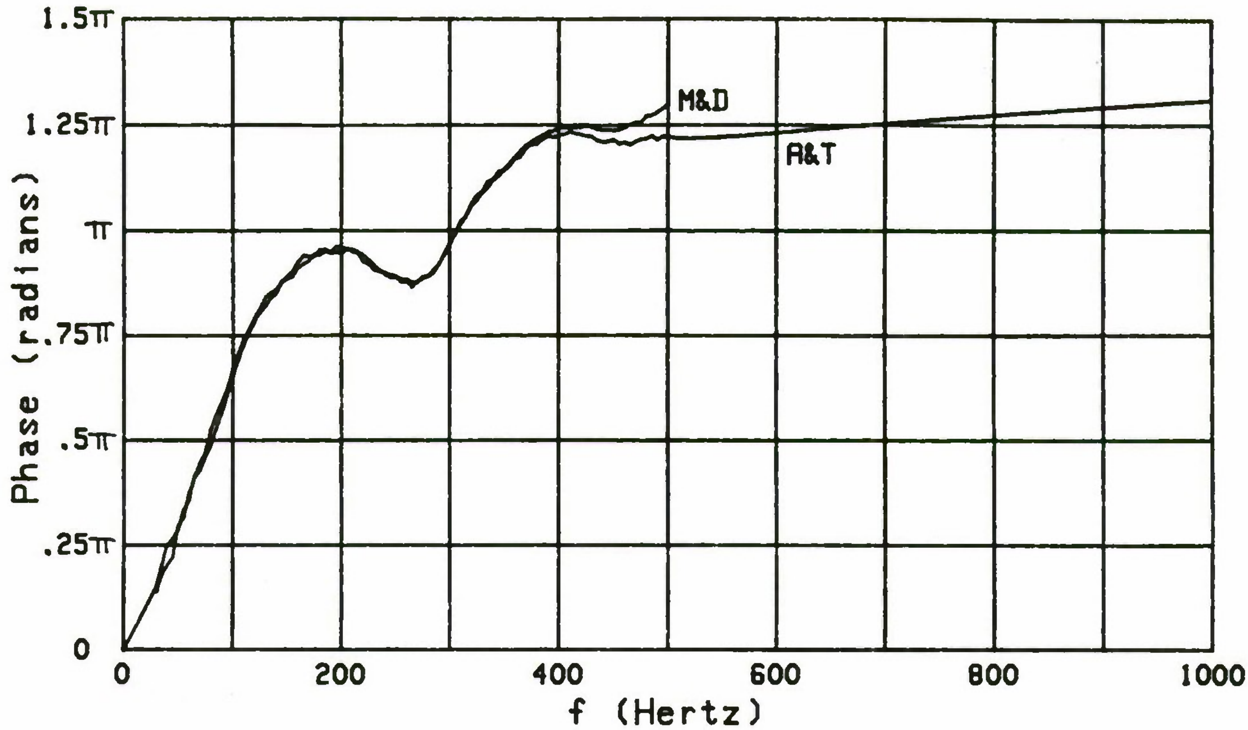

Figure 8. Measured and Transformed Phases for Example B

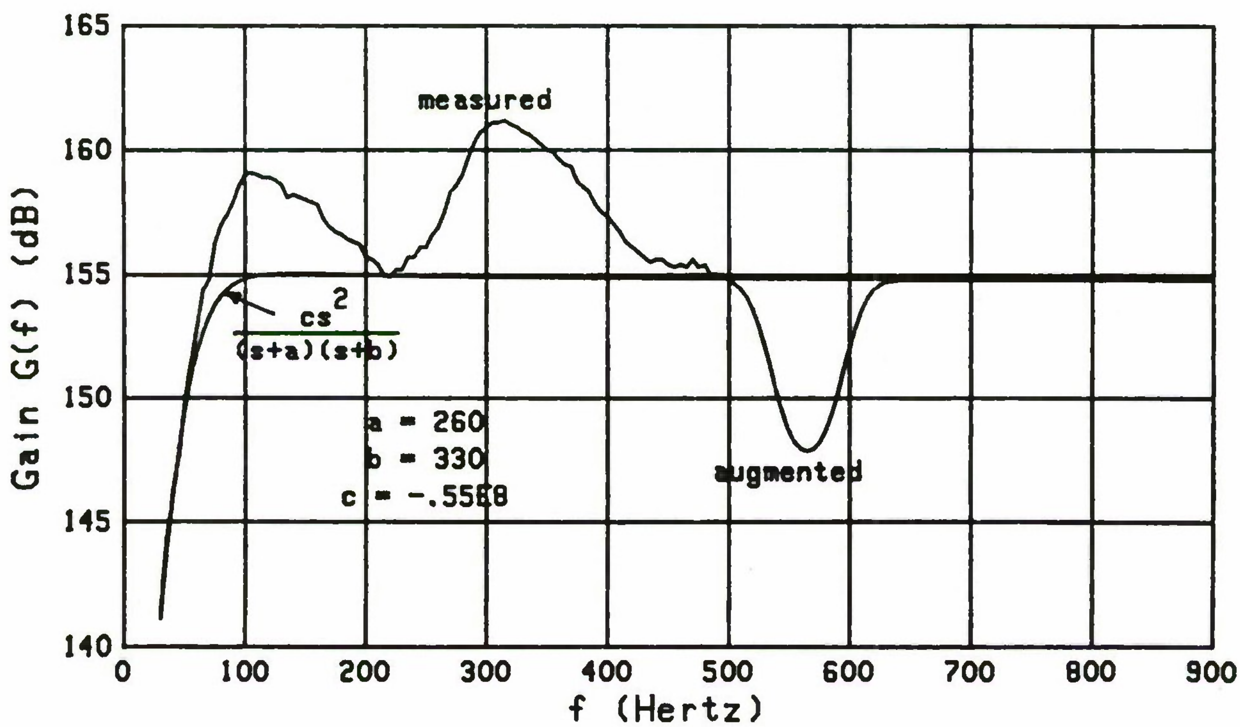

Figure 9. Fitted Gain for Example C

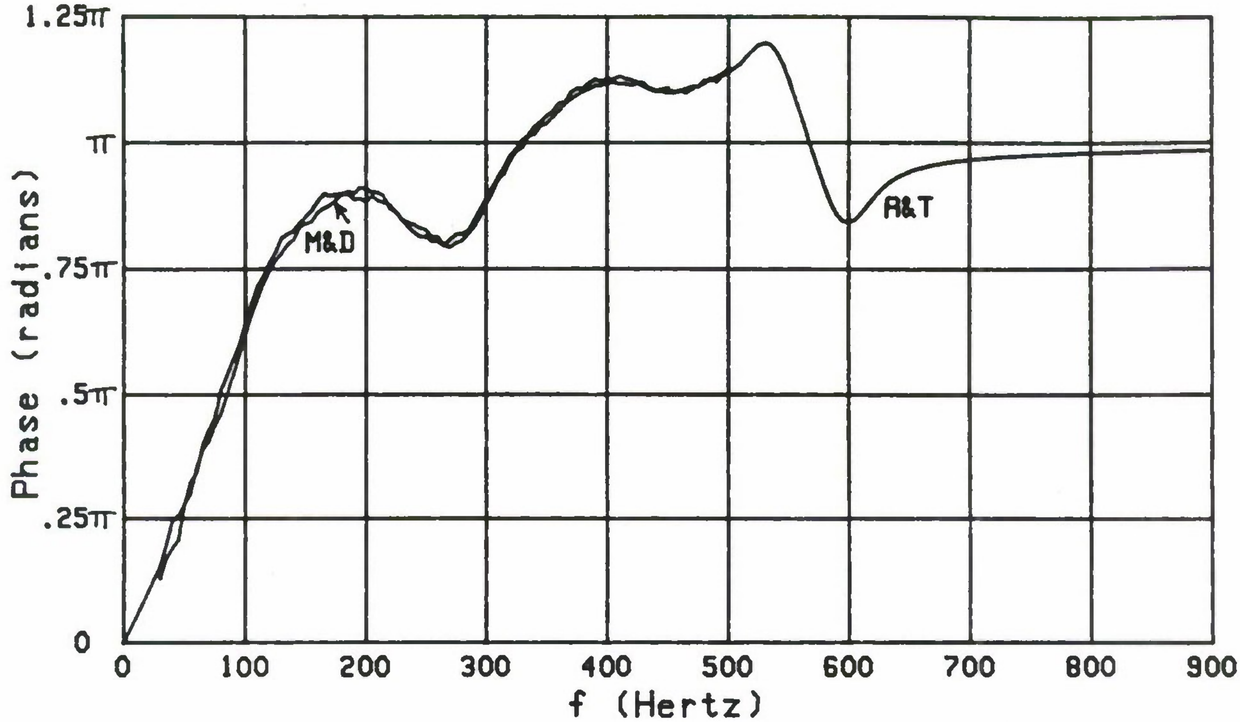

Figure 10. Measured and Transformed Phases for Example C

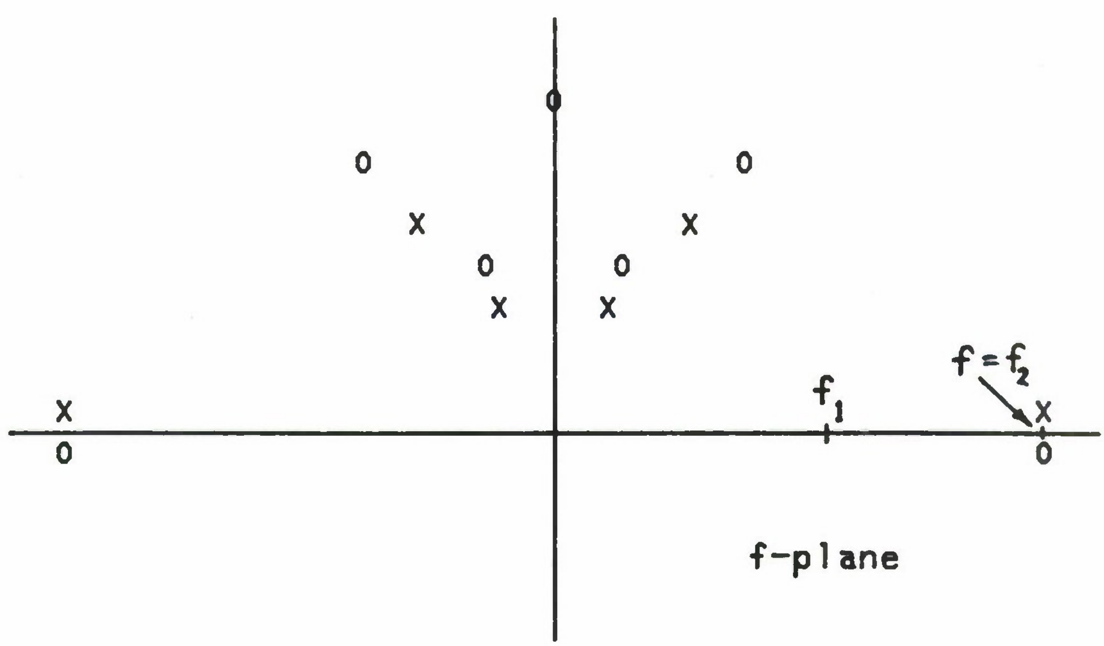

Figure 11. Pole-Zero Locations

SUMMARY

For a minimum-phase filter, the phase shift $\beta(f)$ can be found from the attenuation $\alpha(f)$ by means of two cascaded fast Fourier transforms, once the logarithmic singularities in $\alpha(f)$ have been subtracted out and handled analytically. A partial accuracy check is automatically built into the procedure, because the real part of the output should agree with the given input; the imaginary part of the output is the desired minimum-phase result. This Fourier approach yields the entire phase curve for all frequencies, not just a point-by-point output, as a Hilbert transform numerical integration would give.

In order to use this procedure, the attenuation must be measured for all frequencies, or at least for large enough and small enough frequencies that the asymptotic behavior is well developed and obvious. A plot of the attenuation (or decibel gain) on a logarithmic frequency abscissa is recommended for this purpose, because the filter magnitude characteristic should approach a straight line with a decay equal to a multiple of 6 dB/octave in the neighborhood of zero and infinite frequencies. Failure to make a complete set of measurements will lead to the need for extrapolation and the attendant errors that can occur with such a procedure, as illustrated here. Furthermore, statements about the minimum-phase behavior of a particular filter can only be made (with)in that same frequency range.

APPENDIX A. PRINCIPAL VALUE INTEGRAL EVALUATION

Through a change of variable, a principal value integral can be put in the form

$$I = \int_{-b}^{b} dt \, \frac{g(t)}{t} \,, \quad \text{where } g(0) \neq 0 \,. \tag{A-1}$$

Limit b can be finite or infinite. (For example, (8) fits this form when we let $g(t) = G(x-t)/\pi$.) Although (A-1) is a principal value integral, it can be expressed as (ordinary integrals)

$$I = \int_{-b}^{b} dt \, \frac{g_o(t)}{t} = 2\int_{0}^{b} dt \, \frac{g_o(t)}{t} = \int_{0}^{b} \frac{dt}{t} \, [g(t) - g(-t)] \,, \tag{A-2}$$

where $g_o(t)$ is the odd part of $g(t)$; see definition (5). This form can be used for numerical evaluation whether b is finite or not. If b is infinite, the integrand of the last integral in (A-2) maintains the same decay with t as original integral (A-1). This is not true of the sometimes recommended alternative form

$$I = \int_{-b}^{b} dt \, \frac{g(t) - g(0)}{t} \,, \tag{A-3}$$

which decays very slowly with t, although it is finite at the origin $t = 0$. However, another alternative that advantageously uses this subtraction device is given later in (A-11).

A simple example of (A-1)-(A-2), for b finite, is furnished by the integral

$$I = \int_{-b}^{b} dt\ \frac{\exp(t)}{t} = 2\int_{0}^{b} dt\ \frac{\sinh(t)}{t}\ , \qquad \text{(A-4)}$$

the latter of which has a well-behaved integrand at $t = 0$.

DERIVATIVE EVALUATION

In general, the last integrand in (A-2) behaves as

$$\frac{g(t) - g(-t)}{t} \sim 2\ g'(0) \quad \text{as } t \to 0\ . \qquad \text{(A-5)}$$

Therefore, in order to use (A-2), it is necessary to have $g'(0)$. If all we can easily evaluate is $g(t)$, and not its derivative $g'(0)$, a good approximation is available through the following device. We know that $g'(0)$ is approximated by

$$\frac{g(\varepsilon) - g(-\varepsilon)}{2\varepsilon} \quad \text{for small } \varepsilon\ . \qquad \text{(A-6)}$$

However, if ε is too large, this is a poor approximation, whereas if ε is too small, round-off errors cause numerical stability problems. But we know that

$$\frac{g(\varepsilon) - g(-\varepsilon)}{2\varepsilon} = g'(0) + \frac{1}{6}\ g'''(0)\ \varepsilon^2 + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0\ . \qquad \text{(A-7)}$$

So, letting $F(\varepsilon)$ be the left-hand side of (A-7), we have, to second order,

$$\left.\begin{aligned} F(\varepsilon) &= A_o + A_1\ \varepsilon^2 \\ F(\varepsilon/2) &= A_o + A_1\ \varepsilon^2/4 \end{aligned}\right\} \text{ where } A_o \text{ and } A_1 \text{ are unknown}\ . \qquad \text{(A-8)}$$

The desired unknown follows easily from (A-8) as

$$A_o = \frac{4\ F(\varepsilon/2) - F(\varepsilon)}{3} \approx g'(0)\ . \qquad \text{(A-9)}$$

This procedure is an extrapolation to the limit; it uses $\varepsilon/2$ as the smallest argument of F.

A program for the evaluation of $g'(t)$ at general t is furnished here in BASIC; it requires specification of a tolerance Tol in line 70 of the function subroutine FNDerivl.

```
10  INPUT T
20  Derl=FNDerivl(T)
30  PRINT T,Derl     !  t,g'(t)
40  END
50  !
60  DEF FNDerivl(T)  !  ~g'(t) via extrapolation
70  Tol=1.E-6        !  tolerance
80  E=.2             !  epsilon (start)
90  E=E*.5
100 V1=V2
110 V2=(FNG(T+E)-FNG(T-E))/(2.*E)
120 V=V2+(V2-V1)/3.
130 IF ABS(V2/V-1.)>Tol THEN 90
140 RETURN V
150 FNEND
160 !
170 DEF FNG(T)
180 RETURN EXP(T)    !  example exp(t)
190 FNEND
```

An application of this program to the exp(t) example in line 180, at argument t = 1.1, yielded an error of -7.8E-13.

If we instead kept terms to fourth order in (A-7), an extension to (A-8) yields approximation

$$g'(0) \approx \frac{1}{45}\left[64\ F\left(\frac{\varepsilon}{4}\right) - 20\ F\left(\frac{\varepsilon}{2}\right) + F(\varepsilon)\right]\ . \qquad \text{(A-10)}$$

This procedure uses $\varepsilon/4$ as the smallest argument of F.

AN ALTERNATIVE SUBTRACTION PROCEDURE

We now express (A-1) in the form

$$I = \int_{-b}^{b} dt\, \frac{g(t)}{t} = \int_{-a}^{a} dt\, \frac{g(t)}{t} + \int_{R} dt\, \frac{g(t)}{t}\ , \tag{A-11}$$

where limit a is chosen for convenience and R is the union $(-b,-a) \cup (a,b)$. Then, as done in (A-3),

$$I = \int_{-a}^{a} dt\, \frac{g(t) - g(0)}{t} + \int_{R} dt\, \frac{g(t)}{t}\ . \tag{A-12}$$

These are both ordinary integrals now. The first integrand is finite at $t = 0$, with value $g'(0)$, while the second integrand maintains its original decay as $x \to \pm b$.

SECOND DERIVATIVE EVALUATION

The procedure presented in (A-5)-(A-9), for the approximate evaluation of first derivative $g'(0)$, can be extended to the second derivative $g''(0)$ as follows. We know that

$$\frac{g(\varepsilon) + g(-\varepsilon)}{2} = g(0) + \frac{1}{2}\, g''(0)\, \varepsilon^2 + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0\ . \tag{A-13}$$

Therefore,

$$\frac{g(\varepsilon) + g(-\varepsilon) - 2g(0)}{\varepsilon^2} = g''(0) + O(\varepsilon^2)\ . \tag{A-14}$$

Letting $D(\varepsilon)$ be the left-hand side of (A-14), we have, to second order,

$$\left.\begin{array}{l} D(\varepsilon) = B_o + B_1\,\varepsilon^2 \\ D(\varepsilon/2) = B_o + B_1\,\varepsilon^2/4 \end{array}\right\} \text{ where } B_o \text{ and } B_1 \text{ are unknown .} \tag{A-15}$$

The desired solution is

$$B_o = \frac{4\,D(\varepsilon/2) - D(\varepsilon)}{3} \cong g''(0)\ . \tag{A-16}$$

This is an extrapolation to the limit; it uses $\varepsilon/2$ as the smallest argument of D. A program for the evaluation of g"(t) at general t is given below in BASIC; it requires specification of a tolerance Tol in line 70 of the function subroutine FNDeriv2.

```
10   INPUT T
20   Der2=FNDeriv2(T)
30   PRINT T,Der2     !  t,g"(t)
40   END
50   !
60   DEF FNDeriv2(T)  !  ~g"(t) via extrapolation
70   Tol=1.E-6        !  tolerance
80   E=.2             !  epsilon (start)
90   G2=2.*FNG(T)
100  E=E*.5
110  V1=V2
120  V2=(FNG(T+E)+FNG(T-E)-G2)/(E*E)
130  V=V2+(V2-V1)/3.
140  IF ABS(V2/V-1.)>Tol THEN 100
150  RETURN V
160  FNEND
170  !
180  DEF FNG(T)
190  RETURN EXP(T)    !  example exp(t)
200  FNEND
```

An application of this program to the exp(t) example in line 190, at argument 1.1, yielded an error of 1.6E-11.

APPENDIX B. FOURIER TRANSFORM OF GENERALIZED FUNCTION

We are interested in finding the Fourier transform of the generalized function

$$\frac{\exp(-a\tau)}{\tau}\, U(\tau) \;, \qquad a > 0 \;, \tag{B-1}$$

where $U(\tau)$ is the unit step function. Letting $\omega = 2\pi f$, the integral of interest is

$$I = \int \frac{d\tau}{\tau} \exp(-a\tau)\; U(\tau)\; \exp(-i\omega\tau) =$$

$$= \int \frac{d\tau}{\tau} \left[\exp(-a\tau) - 1 + 1\right] U(\tau)\; \exp(-i\omega\tau) =$$

$$= - \int_0^{+\infty} \frac{d\tau}{\tau} \left[1 - \exp(-a\tau)\right] \exp(-i\omega\tau) + \int \frac{d\tau}{\tau}\, U(\tau)\; \exp(-i\omega\tau) =$$

$$= - \ln\!\left(\frac{a + i\omega}{i\omega}\right) - \left[i\frac{\pi}{2}\, \mathrm{sgn}\!\left(\frac{\omega}{2\pi}\right) + \ln\left|\frac{\omega}{2\pi}\right| + C'\right] = \tag{B-2}$$

$$= - \ln(a + i\omega) + \ln(i\omega) - i\frac{\pi}{2}\, \mathrm{sgn}(\omega) - \ln|\omega| + \ln(2\pi) - C'. \tag{B-3}$$

In (B-2), we used [4; page 334, 3.434 2] and [6; page 43, row 3, column 3, with $m = 1$]. But since

$$\ln(i\omega) = \begin{Bmatrix} i\pi/2 + \ln|\omega| & \text{for } \omega > 0 \\ -i\pi/2 + \ln|\omega| & \text{for } \omega < 0 \end{Bmatrix} + i2\pi n =$$

$$= i\frac{\pi}{2}\, \mathrm{sgn}(\omega) + \ln|\omega| + i2\pi n \;, \quad n \text{ integer} \;, \tag{B-4}$$

we can express (B-3) as

$$I = -\ln(a + i\omega) + C , \quad \text{where } C = \ln(2\pi) - C' + i2\pi n . \qquad \text{(B-5)}$$

Thus, we have the Fourier transform pair

$$\frac{\exp(-a\tau)}{\tau} U(\tau) \longleftrightarrow -\ln(a + i2\pi f) + C , \qquad \text{(B-6)}$$

where C is an arbitrary constant. The reason for the presence of C is that the generalized function $\frac{1}{\tau} U(\tau)$ is indeterminate within an additive arbitrary multiple of the delta function $\delta(\tau)$.

For the example in (33) of $H(f) = 1/(1 + i2\pi f)$, we have $Q(f) = \ln(1 + i2\pi f)$. Application of pair (B-6), with $a = 1$, to (39) then yields causal function

$$q(\tau) = -\frac{\exp(-\tau)}{\tau} U(\tau) . \qquad \text{(B-7)}$$

APPENDIX C. HILBERT TRANSFORM MANIPULATION

It was noted below (45) that the Hilbert transform of attenuation $\alpha(f)$ encounters integrals with logarithmic infinities and must be handled more carefully. This problem is treated in [3; pages 206-8], by dividing the attenuation by a factor that is quadratic in f, rather than linear. In current notation, that result is [3; (10-67)]

$$\beta(f) = \frac{f}{\pi} \int_{-\infty}^{+\infty} du \, \frac{\alpha(u)}{u^2 - f^2} \, . \tag{C-1}$$

If we utilize the property employed in [3; page 208, line 2], namely that attenuation $\alpha(f)$ is even, we can develop (C-1) as

$$\beta(f) = \frac{2f}{\pi} \int_{0}^{+\infty} du \, \frac{\alpha(u)}{u^2 - f^2} =$$

$$= - \frac{1}{\pi} \int_{0}^{+\infty} du \, \alpha(u) \left(\frac{1}{f - u} + \frac{1}{f + u} \right) = \tag{C-2}$$

$$= - \frac{1}{\pi} \int_{0}^{+\infty} du \, \frac{\alpha(u)}{f - u} - \frac{1}{\pi} \int_{0}^{+\infty} du \, \frac{\alpha(u)}{f + u} = \tag{C-3}$$

$$= - \frac{1}{\pi} \int_{0}^{+\infty} du \, \frac{\alpha(u)}{f - u} - \frac{1}{\pi} \int_{-\infty}^{0} dv \, \frac{\alpha(-v)}{f - v} =$$

$$= - \frac{1}{\pi} \int_{-\infty}^{+\infty} du \, \frac{\alpha(u)}{f - u} = - \underline{H}\{\alpha(f)\} \, . \tag{C-4}$$

The step leading from (C-2) to (C-3) presumes that both of the latter integrals converge separately, which need not be the case for attenuations $\alpha(f)$; this is the reason for the quadratic denominator adopted in (C-1), which guaranteed convergence of that integral.

Rather than using Hilbert transforms and having to employ the method of (C-1), we have resorted instead to the use of Fourier transforms, as outlined in (46). Of course, a similar problem arises there, as mentioned in the sequel to (51). The method of circumventing the difficulty, in the Fourier approach, is to subtract out the singularities and handle them analytically, as described in (54)-(57).

The justification of this procedure, using modified Hilbert transform (C-1) as a starting point, is as follows. Express given attenuation $\alpha(f)$ in two parts, as in (54), where residue $\alpha_2(f)$ has a convergent Hilbert transform integral

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} du\, \frac{\alpha_2(u)}{f-u} = \underline{H}\{\alpha_2(f)\} \quad \text{for all } f . \tag{C-5}$$

The phase shift $\beta(f)$ corresponding to attenuation $\alpha(f)$ is then given by sum (56), where, following (C-1),

$$\beta_1(f) = \frac{f}{\pi}\int_{-\infty}^{+\infty} du\, \frac{\alpha_1(u)}{u^2-f^2} \tag{C-6}$$

and $\beta_2(f)$ is available as the negative of (C-5). The proof of this last claim follows immediately from the derivation in (C-1)-(C-4) if we replace $\alpha(f)$ and $\beta(f)$ everywhere by $\alpha_2(f)$ and

$\beta_2(f)$, respectively. This is legitimate because the existence of (C-5) for residual attenuation $\alpha_2(f)$ now allows the separation into two convergent integrals, as done in (C-3).

We do not actually use (C-5) or (C-6). Instead, (C-6) is accomplished by using known closed form attenuation/minimum-phase pairs for $\alpha_1(f)$ and $\beta_1(f)$, while (C-5) is replaced by the Fourier approach given in (46), with $\alpha_2(f)$ and $\beta_2(f)$ substituted for $\alpha(f)$ and $\beta(f)$, respectively. The inverse Fourier transform integral in the top line of (46), but now in terms of $\alpha_2(f)$, is convergent.

(For interest, an example of the application of (C-6) is afforded by attenuation-phase pair (51). This fact is immediately verified by use of [4; 4.295 8].)

APPENDIX D. EXAMPLES OF ATTENUATION/MINIMUM-PHASE PAIRS

In this appendix, we list a few attenuation/minimum-phase pairs that can be used in the subtraction procedure presented in (54)-(57) to eliminate the divergent integrands encountered. For convenience of notation, we employ the Laplace transform of the impulse response, namely

$$L(s) = \int_{0}^{+\infty} d\tau \, \exp(-s\tau) \, h(\tau) \; , \tag{D-1}$$

where we have specifically limited consideration to causal filters. The connection with the Fourier transform (1) is

$$H(f) = L(i2\pi f) \; . \tag{D-2}$$

In the following, a, b, and c are real positive constants, and $\omega = 2\pi f$.

EXAMPLE 1:

$$L(s) = \frac{c}{s + a} \; ,$$

$$\alpha(f) = \frac{1}{2} \ln(a^2 + \omega^2) - \ln(c) \; , \quad \beta(f) = \arctan(\omega/a) \; . \tag{D-3}$$

In the limit as $a \to 0+$,

$$\alpha(f) = \ln|\omega| - \ln(c) \; , \quad \beta(f) = \frac{\pi}{2} \operatorname{sgn}(\omega) \; . \tag{D-4}$$

EXAMPLE 2:

$$L(s) = \frac{c\,s}{s + a} ,$$

$$\alpha(f) = \frac{1}{2} \ln(a^2 + \omega^2) - \ln|\omega| - \ln(c) ,$$

$$\beta(f) = \arctan(\omega/a) - \frac{\pi}{2} \operatorname{sgn}(\omega) . \tag{D-5}$$

EXAMPLE 3:

$$L(s) = \frac{c\,s}{(s + a)(s + b)} ,$$

$$\alpha(f) = \frac{1}{2} \ln(a^2 + \omega^2) + \frac{1}{2} \ln(b^2 + \omega^2) - \ln|\omega| - \ln(c) ,$$

$$\beta(f) = \arctan(\omega/a) + \arctan(\omega/b) - \frac{\pi}{2} \operatorname{sgn}(\omega) . \tag{D-6}$$

This attenuation reaches a minimum at $\omega = (ab)^{1/2}$, at which point the phase goes through zero.

EXAMPLE 4:

$$L(s) = \frac{c}{(s + a)^2 + b^2} ,$$

$$\alpha(f) = \frac{1}{2} \ln\left[(a^2 + (\omega + b)^2\right] + \frac{1}{2} \ln\left[a^2 + (\omega - b)^2\right] - \ln(c) ,$$

$$\beta(f) = \arctan\left(\frac{\omega - b}{a}\right) + \arctan\left(\frac{\omega + b}{a}\right) . \tag{D-7}$$

APPENDIX E. NUMERICAL EVALUATION OF (46)

We repeat here the cascaded Fourier transform operations listed in (46):

$$\underline{g}(\tau) \equiv \underline{F}^{-1}\{\alpha(f)\} , \tag{E-1}$$

$$q(\tau) = 2\, \underline{g}(\tau)\, U(\tau) , \tag{E-2}$$

$$\alpha(f) + i\, \beta(f) = \underline{F}\{q(\tau)\} . \tag{E-3}$$

We limit consideration to the case where attenuation $\alpha(f)$ is even, which is the typical practical situation. Also, we weight the inverse Fourier transform in (E-1) by real symmetric window $W(f)$, which is zero for $|f| > M\Delta$. We then get approximation

$$\underline{g}_a(\tau) \equiv \int_{-\infty}^{+\infty} df\, \exp(i2\pi f\tau)\, \alpha(f)\, W(f) =$$

$$= 2\,\mathrm{Re} \int_{0}^{+\infty} df\, \exp(-i2\pi f\tau)\, \alpha(f)\, W(f) =$$

$$= 2\,\mathrm{Re} \int_{0}^{M\Delta} df\, \exp(-i2\pi f\tau)\, \alpha(f)\, W(f) =$$

$$\cong 2\,\mathrm{Re} \sum_{n=0}^{M} s_n\, \Delta\, \exp(-i2\pi n\Delta\tau)\, \alpha(n\Delta)\, W(n\Delta) \equiv \underline{g}_b(\tau) , \tag{E-4}$$

where we sample in frequency f with increment Δ. We also use some integration rule like trapezoidal or Simpson; for example, the trapezoidal rule has $s_n = 1$, except for $s_o = s_M = 1/2$.

The approximation $g_b(\tau)$, defined by the bottom line of (E-4), has period $1/\Delta$ in τ. Therefore, we compute it at the points

$$\tau = \frac{m}{N\Delta} \quad \text{for } 0 \leq m \leq N - 1 \;, \tag{E-5}$$

which cover a full period of $g_b(\tau)$. There follows

$$g_b\left(\frac{m}{N\Delta}\right) = 2\Delta \; \mathrm{Re} \sum_{n=0}^{M} s_n \exp(-i2\pi nm/N) \; \alpha(n\Delta) \; W(n\Delta) \;, \tag{E-6}$$

which is an N-size fast Fourier transform of M + 1 data points. Any surplus points can be collapsed, if desired, without loss of accuracy; see [7; pages 4-5], for example.

Operations (E-2) and (E-3) can be combined to read

$$Q(f) = \alpha(f) + i\;\beta(f) = 2 \int_0^{+\infty} d\tau \exp(-i2\pi f\tau) \; g(\tau) \;. \tag{E-7}$$

Because all we have available is approximation $g_b(\tau)$ from (E-4), we adopt the following approximation to Q(f), based on (E-7):

$$Q_a(f) \equiv 2 \int_0^{+\infty} d\tau \exp(-i2\pi f\tau) \; g_b(\tau) =$$

$$\cong 2 \int_0^{.5/\Delta} d\tau \exp(-i2\pi f\tau) \; g_b(\tau) = \tag{E-8}$$

$$\cong 2 \sum_{m=0}^{N/2} w_m \frac{1}{N\Delta} \exp\left(-i2\pi f\frac{m}{N\Delta}\right) g_b\left(\frac{m}{N\Delta}\right) \equiv Q_b(f) \;, \tag{E-9}$$

where w_m is an integration weight. The integral in (E-8) was limited to $.5/\Delta$ in τ, because approximation $\underline{q}_b(\tau)$ in (E-4) is only available up to that limit without aliasing.

The period of the final approximation $Q_b(f)$ in (E-9) is $N\Delta$ in f. Therefore, we limit its computation to the values

$$Q_b(n\Delta) = \frac{2}{N\Delta} \sum_{m=0}^{N/2} w_m \exp(-i2\pi nm/N)\, \underline{q}_b\left(\frac{m}{N\Delta}\right) \quad \text{for } 0 \le n \le N-1 \ . \quad \text{(E-10)}$$

This can be accomplished as an N-size fast Fourier transform of $N/2 + 1$ data points. The final approximation to desired phase $\beta(f)$ in (E-7) is available as the imaginary part of (E-10), at frequencies $f = n\Delta$. In addition, the real part of (E-10) should be in very good agreement with specified attenuation values $\{\alpha(n\Delta)\ W(n\Delta)\}$ used in (E-6); this serves as an accuracy check on the complete procedure. Equations (E-6) and (E-10) are the final results. Strictly, (E-6) should be applied only to the residual attenuation $\alpha_2(f)$ defined in (55); then (E-10) furnishes an approximation to $\alpha_2(f) + i\ \beta_2(f)$. A program in BASIC for the Hewlett Packard 9000 computer, for the procedure given above, is presented below.

```
10 !  NUSC TR 8667, FOURIER PROCEDURE APPLIED
20 !  TO REAL EVEN FUNCTION OF FREQUENCY
30    Deltaf=5.                          !  SAMPLING INCREMENT IN FREQUENCY
40    Fmax=900.                          !  MAXIMUM FREQUENCY
50    N=16384                            !  SIZE OF FFT
60    A=260.                             !  FILTER PARAMETERS
70    B=330.                             !  FOR
80    C=-.55E8                           !  EXAMPLE C
90    COM A,B,C
100   REDIM Cos(0:N/4),X(0:N-1),Y(0:N-1)
110   DIM Cos(4096),X(16384),Y(16384),Realeven(25000),Phase(6:100)
120   DOUBLE N,M,Ns,Ms,N2,M2             !  INTEGERS
130   T=2.*PI/N
140   FOR Ns=0 TO N/4
150   Cos(Ns)=COS(T*Ns)                  !  QUARTER-COSINE TABLE
160   NEXT Ns
170   M=Fmax/Deltaf
180   REDIM Realeven(0:M)
190   CALL Input_real_even(Deltaf,Fmax,Realeven(*)) !  RESIDUAL
200   MAT X=(0.)                         !  ATTENUATION ALPHA2
210   MAT Y=(0.)
220   X(0)=.5*Realeven(0)
230   Ms=M MODULO N
240   X(Ms)=.5*Realeven(M)
250   FOR Ns=1 TO M-1
260   Ms=Ns MODULO N                     !  COLLAPSING
270   X(Ms)=X(Ms)+Realeven(Ns)
280   NEXT Ns
290   CALL Fft14(N,Cos(*),X(*),Y(*))     !  FOURIER TRANSFORM
300   N2=N/2                             !  INTO TIME DOMAIN
310     GINIT
320     PLOTTER IS "GRAPHICS"
330     GRAPHICS ON
340     WINDOW -N2,N2,-6,2
350     LINE TYPE 3
360     GRID N/8,1
370     PRINT "FOURIER TRANSFORM (TIME DOMAIN)"
380     FOR Ns=-N2 TO N2
390     Ms=Ns MODULO N
400     PLOT Ns,LGT(ABS(X(Ms))+1.E-99)   !  TIME DOMAIN FUNCTION
410     NEXT Ns
420     PENUP
430     PAUSE
440    MAT Y=(0.)
450    T=4./N                            !  2 Deltaf * 2 / (N Deltaf)
460    FOR Ms=0 TO N2
470    X(Ms)=X(Ms)*T                     !  DOUBLE FOR POSITIVE TIME
480    NEXT Ms
490    X(0)=X(0)*.5
500    X(N2)=X(N2)*.5
510    FOR Ms=N2+1 TO N-1
520    X(Ms)=0.                          !  ZERO FOR NEGATIVE TIME
530    NEXT Ms
540    CALL Fft14(N,Cos(*),X(*),Y(*))    !  FOURIER TRANSFORM
550      M2=M*2                          !  INTO FREQUENCY DOMAIN
```

```
560     GCLEAR
570     WINDOW 0,M2,-1,1
580     LINE TYPE 3
590     GRID N/16,.2
600     PRINT "ORIGINAL INPUT (FREQUENCY DOMAIN)"
610     FOR Ns=0 TO MIN(M,N2)
620     PLOT Ns,Realeven(Ns)            !  ORIGINAL INPUT
630     NEXT Ns
640     PENUP
650     PAUSE
660     LINE TYPE 1
670     FOR Ns=0 TO M2
680     PLOT Ns,X(Ns)                   !  F-T-F APPROXIMATION
690     NEXT Ns
700     PENUP
710     PAUSE
720   DATA -38.6,-48.2,-54.8,-60.4,-76.2,-82.1,-94.5,-103.8,-109.1,-117.1
730   DATA -124.1,-134.0,-143.1,-152.9,-163.1,-172.4,179.1,171.1,164.2,157.9
740   DATA 152.8,147.1,142.8,135.8,131.9,128.7,122.8,118.7,115.1,110.6
750   DATA 105.9,103.4,102.8,99.9,98.6,93.8,93.1,91.2,89.6,89.5
760   DATA 89.6,89.6,89.2,88.1,85.6,84.5,82.0,81.1,79.0,74.7
770   DATA 71.4,66.5,61.3,55.1,48.1,41.6,34.0,29.3,22.0,16.1
780   DATA 12.2,5.7,2.4,-3.1,-6.5,-11.3,-16.2,-21.2,-25.7,-29.7
790   DATA -33.4,-37.0,-40.7,-43.5,-47.0,-49.5,-51.6,-54.1,-56.2,-59.4
800   DATA -61.0,-62.4,-64.2,-66.7,-68.7,-71.4,-74.6,-78.1,-81.4,-83.8
810   DATA -88.7,-91.3,-95.0,-98.7,-103.1
820   READ Phase(*)                     !  MEASURED PHASE IN DEGREES
830   FOR Ns=22 TO 100
840   Phase(Ns)=Phase(Ns)-360.          !  UN-WRAPPING OF PHASE
850   NEXT Ns
860   MAT Phase=Phase*(-PI/180.)        !  MEASURED PHASE IN RADIANS
870     T=2.*PI*Deltaf
880     FOR Ns=0 TO N2
890     W=T*Ns
900     Phaseapp=ATN((W-B)/A)+ATN((W+B)/A) ! PHASE BETA1 OF APPROX.
910     X(Ns)=Phaseapp+Y(Ns)            !  CALCULATED PHASE IN RADIANS:
920     NEXT Ns                         !  BETA = BETA1 + BETA2
930     GCLEAR
940     WINDOW 0,180,0,PI*1.25
950     LINE TYPE 1
960     GRID 20,PI*.25
970     PRINT "PHASE (FREQUENCY DOMAIN)"
980     FOR Ns=0 TO 180
990     PLOT Ns,X(Ns)                   !  PHASE VIA FOURIER PROCEDURE
1000    NEXT Ns
1010    PENUP
1020    LINE TYPE 3
1030    FOR Ns=6 TO 100
1040    PLOT Ns,Phase(Ns)-Ns*.0448      !  MEASURED PHASE WITH
1050    NEXT Ns                         !  TIME DELAY CORRECTION
1060    PENUP                           !  OF 1.43 MILLISECONDS
1070    PAUSE
1080  END
1090  !
```

```
1100    SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1110    DOUBLE Log2n,N1,N2,N3,N4,J,K    !  INTEGERS < 2^31 = 2,147,483,648
1120    DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1130    IF N=1 THEN SUBEXIT
1140    IF N>2 THEN 1220
1150    A=X(0)+X(1)
1160    X(1)=X(0)-X(1)
1170    X(0)=A
1180    A=Y(0)+Y(1)
1190    Y(1)=Y(0)-Y(1)
1200    Y(0)=A
1210    SUBEXIT
1220    A=LOG(N)/LOG(2.)
1230    Log2n=A
1240    IF ABS(A-Log2n)<1.E-8 THEN 1270
1250    PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
1260    PAUSE
1270    N1=N/4
1280    N2=N1+1
1290    N3=N2+1
1300    N4=N3+N1
1310    FOR I1=1 TO Log2n
1320    I2=2^(Log2n-I1)
1330    I3=2*I2
1340    I4=N/I3
1350    FOR I5=1 TO I2
1360    I6=(I5-1)*I4+1
1370    IF I6<=N2 THEN 1410
1380    A1=-Cos(N4-I6-1)
1390    A2=-Cos(I6-N1-1)
1400    GOTO 1430
1410    A1=Cos(I6-1)
1420    A2=-Cos(N3-I6-1)
1430    FOR I7=0 TO N-I3 STEP I3
1440    I8=I7+I5-1
1450    I9=I8+I2
1460    T1=X(I8)
1470    T2=X(I9)
1480    T3=Y(I8)
1490    T4=Y(I9)
1500    A3=T1-T2
1510    A4=T3-T4
1520    X(I8)=T1+T2
1530    Y(I8)=T3+T4
1540    X(I9)=A1*A3-A2*A4
1550    Y(I9)=A1*A4+A2*A3
1560    NEXT I7
1570    NEXT I5
1580    NEXT I1
```

```
1590    I1=Log2n+1
1600    FOR I2=1 TO 14
1610    L(I2-1)=1
1620    IF I2>Log2n THEN 1640
1630    L(I2-1)=2^(I1-I2)
1640    NEXT I2
1650    K=0
1660    FOR I1=1 TO L(13)
1670    FOR I2=I1 TO L(12) STEP L(13)
1680    FOR I3=I2 TO L(11) STEP L(12)
1690    FOR I4=I3 TO L(10) STEP L(11)
1700    FOR I5=I4 TO L(9) STEP L(10)
1710    FOR I6=I5 TO L(8) STEP L(9)
1720    FOR I7=I6 TO L(7) STEP L(8)
1730    FOR I8=I7 TO L(6) STEP L(7)
1740    FOR I9=I8 TO L(5) STEP L(6)
1750    FOR I10=I9 TO L(4) STEP L(5)
1760    FOR I11=I10 TO L(3) STEP L(4)
1770    FOR I12=I11 TO L(2) STEP L(3)
1780    FOR I13=I12 TO L(1) STEP L(2)
1790    FOR I14=I13 TO L(0) STEP L(1)
1800    J=I14-1
1810    IF K>J THEN 1880
1820    A=X(K)
1830    X(K)=X(J)
1840    X(J)=A
1850    A=Y(K)
1860    Y(K)=Y(J)
1870    Y(J)=A
1880    K=K+1
1890    NEXT I14
1900    NEXT I13
1910    NEXT I12
1920    NEXT I11
1930    NEXT I10
1940    NEXT I9
1950    NEXT I8
1960    NEXT I7
1970    NEXT I6
1980    NEXT I5
1990    NEXT I4
2000    NEXT I3
2010    NEXT I2
2020    NEXT I1
2030    SUBEND
2040    !
```

```
2050    SUB Input_real_even(Deltaf,Fmax,Realeven(*))
2060    DOUBLE Ns
2070    ALLOCATE Db(6:180)                    !  30:900 HZ
2080    DATA 41.3,44.3,46.1,47.6,49.9,51.4,52.9,54.4,54.8,56.3
2090    DATA 57.0,57.4,57.9,58.6,59.0,59.1,59.0,58.9,58.9,58.8
2100    DATA 58.6,58.1,58.2,58.1,58.0,57.9,57.8,57.2,56.9,56.7
2110    DATA 56.6,56.4,56.3,56.2,55.7,55.6,55.4,55.0,54.9,55.2
2120    DATA 55.2,55.7,55.7,56.1,56.1,56.6,56.9,57.5,58.3,58.6
2130    DATA 59.0,59.7,60.3,60.7,60.9,61.1,61.1,61.2,61.0,60.9
2140    DATA 60.7,60.6,60.4,60.2,60.0,59.9,59.6,59.4,59.3,58.7
2150    DATA 58.5,58.3,57.8,57.5,57.3,57.0,56.7,56.3,56.1,55.9
2160    DATA 55.7,55.5,55.6,55.6,55.4,55.3,55.4,55.3,55.6,55.3
2170    DATA 55.4,55.0,55.0,55.0,54.8
2180    REDIM Db(6:100)
2190    READ Db(*)
2200    MAT Db=Db+(100.)                      !  MEASURED DB GAIN
2210    REDIM Db(6:180)
2220    FOR Ns=101 TO 180                     !  AUGMENTED DB GAIN
2230    F=Deltaf*Ns
2240    T1=(F-550.)*.04
2250    T2=(F-580.)*.04
2260    Db(Ns)=154.8-5.*EXP(-T1*T1)-5.*EXP(-T2*T2)
2270    NEXT Ns
2280    MAT Realeven=(0.)
2290    COM A,B,C
2300    A2=A*A
2310    B2=B*B
2320    C2=C*C
2330    D1=(A2+B2)*(A2+B2)
2340    D2=2.*(A2-B2)
2350    T=2.*PI*Deltaf
2360    FOR Ns=6 TO 180
2370    W=T*Ns
2380    W2=W*W
2390    W4=W2*W2
2400    P=C2*W4/(D1+D2*W2+W4)
2410    Attenapp=-.5*LOG(P)                   !  APPROX. ATTEN. ALPHA1
2420    Atten=Db(Ns)/(-8.686)                 !  ATTENUATION ALPHA
2430    Realeven(Ns)=Atten-Attenapp           !  RESIDUAL ATTEN. ALPHA2
2440    NEXT Ns
2450    SUBEND
```

REFERENCES

1. N. Balabanian and W. R. LePage, "What is a Minimum-Phase Network," **Proceedings AIEE**, Communications and Electronics, vol. 75, no. 22, pages 785-787, January 1956.

2. E. A. Guillemin, **The Mathematics of Circuit Analysis**, John Wiley & Sons, Inc., New York, NY, 1951.

3. A. Papoulis, **The Fourier Integral and Its Applications**, McGraw-Hill Book Company, Inc., New York, NY, 1962.

4. I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, Inc., New York, NY, 1980.

5. A. Papoulis, **Signal Analysis**, McGraw-Hill Book Company, Inc., New York, NY, 1977.

6. M. J. Lighthill, **Introduction to Fourier Analysis and Generalised Functions**, Cambridge University Press, New York, NY, 1964.

7. A. H. Nuttall, **Alias-Free Wigner Distribution Function and Complex Ambiguity Function for Discrete-Time Samples**, NUSC Technical Report 8533, Naval Underwater Systems Center, New London, CT, 14 April 1989.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
Admiralty Research Establishment, England	
Dr. L. J. Lloyd	1
Admiralty Underwater Weapons Establishment, England	
Library	1
Center for Naval Analyses	1
Coast Guard Academy	
Prof. J. J. Wolcin	1
David Taylor Research Center, Bethesda	1
David Taylor Research Center, Annapolis	
P. Prendergast (Code 2744)	1
Defence Research Center, Australia	
Library	1
Defence Research Establishment Atlantic, Nova Scotia	
B. E. Mackey (Library)	1
Defence Research Establishment Pacific, British Columbia	
Dr. D. J. Thomson	1
Defence Science Establishment, HMNZ Dockyard, New Zealand	
Director	1
Defence Scientific Establishment, New Zealand	
Dr. L. H. Hall	1
Defense Advanced Research Projects Agency	
Commanding Officer	1
A. W. Ellinthorpe	1
Defense Intelligence Agency	1
Defense Technical Information Center	12
Dept. of Science & Industrial Research, New Zealand	
M. A. Poletti	1
National Radio Astronomy Observatory	
F. Schwab	1
National Security Agency	
Dr. J. R. Maar (R51)	1
Naval Air Development Center	1
Naval Air Systems Command	
NAIR-93	1
Naval Coastal Systems Center	1
Naval Environmental Prediction Research Facility	1
Naval Intelligence Command	1
Naval Oceanographic Office	1
Naval Ocean Research & Development Activity, NSTL, MS	
Commanding Officer	1
R. Wagstaff (Code 345)	1
Naval Ocean Systems Center, Hawaii	1
Naval Ocean Systems Center, San Diego	
Commanding Officer	1
J. M. Alsup (Code 635)	1
F. J. Harris	1
Naval Personnel Research & Development Center	1
Naval Postgraduate School	
Superintendent	2
Prof. C. W. Therrien (Code 62 TI)	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Naval Research Laboratory, Orlando, USRD	1
Naval Research Laboratory, Washington	
Commanding Officer	1
Dr. P. B. Abraham (Code 5131)	1
W. F. Gabriel (Code 5370)	1
A. A. Gerlach	1
N. Yen (Code 5130)	1
Naval Sea Systems Command	
SEA-00; -63; -63D; -63X; -92R; PMS-402	6
Naval Surface Weapons Center, Dahlgren, VA	1
Naval Surface Weapons Center, White Oak Lab.	1
Naval Surface Weapons Center, Fort Lauderdale	1
Naval Technical Intelligence Center	2
Naval Underwater Systems Center, West Palm Beach	
Officer-in-Charge	1
Dr. R. M. Kennedy (Code 3802)	1
Naval Weapons Center	1
Norwegian Defence Research Establishment	
Dr. J. Glattetre	1
Office of the Chief of Naval Research, Arlington, VA	
OCNR-00; -10; -11; -12; -13; -20; -21; -22; -23 (3)	11
N. L. Gerr (Code 1111)	1
SACLANT Undersea Research Center	
Dr. J. Ianniello	1
Dr. S. Stergiopolous	1
Prof. G. Tacconi	1
Library	1
Sonar and Surveillance Group, Australia	1
Space & Naval Warfare System Cmd.	
SPAWAR-00; -04; -005; PD-80; PMW-181	5
U.S. Air Force, Alabama	
Air University Library	1
U.S. Coast Guard Research & Development Center	
Library	1
Weapons Systems Research Laboratory, Australia	
Signal Processing and Class. Group	1
Underwater Detection Group	1
Brown University	
Documents Library	1
Canberra College of Advanced Education	
P. Morgan	1
Concordia University, Quebec	
Prof. J. Krolik	1
Dalhousie University	
Dr. B. Ruddick	1
Drexel University	
Prof. S. Kesler	1
Harvard University	
Gordon McKay Library	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Indian Institute of Technology, India	
Dr. K. M. M. Prabhu	1
Johns Hopkins University, Applied Physics Laboratory	
Director	1
J. C. Stapleton	1
Lawrence Livermore National Laboratory	1
Los Alamos National Laboratory	1
Marine Biological Laboratory, Woods Hole	1
Marine Physical Laboratory, Scripps	1
Massachusetts Institute of Technology	
Prof. A. Baggaroer	1
Barker Engineering Library	1
Northeastern University	
Prof. C. L. Nikias	1
Penn State University, Applied Research Laboratory	
Director	1
F. W. Symons	1
Purdue University	
Prof. N. Srinivasa	1
Royal Military College of Canada	
Prof. Y. T. Chan	1
Rutgers University	
Prof. S. Orfanidis	1
Sandia National Laboratory	
Director	1
J. Claasen (315)	1
Simon Fraser University	
Dr. E. F. Velez	1
Southeastern Massachusetts University	
Prof. C. H. Chen	1
State University of New York	
Prof. M. Barkat	1
Tel-Aviv University, Israel	
Prof. E. Weinstein	1
United Engineering Center	
Engineering Societies Library	1
University of Alberta, Canada	
K. L. Yeung	1
University of Auckland	
Dr. M. D. Johns	1
University of California, San Diego	
Prof. C. W. Helstrom	1
University of Colorado	
Prof. L. Scharf	1
University of Connecticut	
Prof. C. H. Knapp	1
Wilbur Cross Library	1
University of Florida	
Prof. D. C. Childers	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
University of Illinois	
Dr. D. L. Jones	1
University of Michigan	
Communications & Signal Processing Laboratory	1
W. J. Williams	1
University of Minnesota	
Prof. M. Kaveh	1
University of Queensland	
Dr. B. Boashash	1
University of Rhode Island	
Prof. G. F. Boudreaux-Bartels	1
Prof. S. Kay	1
Prof. D. Tufts	1
Library	1
University of Rochester	
Prof. E. Titlebaum	1
University of Southern California	
Prof. W. C. Lindsey	1
Dr. A. Polydoros	1
University of Strathclyde, Scotland	
Prof. T. S. Durrani	1
University of Technology, England	
Prof. J. W. R. Griffiths	1
University of Texas, Applied Research Laboratory	1
University of Washington	
Applied Physics Laboratory	1
Prof. D. W. Lytle	1
Dr. R. C. Spindel	1
Villanova University	
Prof. M. G. Amin	1
Woods Hole Oceanographic Institution	
Director	1
Dr. E. Weinstein	1
Yale University	
Prof. A. Nehorai	1
Prof. P. M. Schultheiss	1
Prof. F. Tuteur	1
Kline Science Library	1
Applied Seismic Group	
R. Lacoss	1
Astron Research and Engineering	
Dr. A. G. Piersol	1
A&T, North Stonington, CT	
H. Jarvis	1
BB&N, Arlington, VA	
Dr. H. Cox	1
BB&N, Cambridge, MA	
H. Gish	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
BB&N, New London, CT	
Dr. P. G. Cable	1
Bell Communications Research	
J. F. Kaiser	1
D. Sunday	1
Cogent Systems, Inc.	
J. P. Costas	1
Defense Systems, Inc.	
Dr. G. S. Sebestyen	1
Diagnostic/Retrieval Systems, Inc.	
J. Williams	1
EDO Corporation	
M. Blanchard	1
EG&G	
D. Frohman	1
General Electric Company, Moorestown, NJ	
Dr. M. R. Allen	1
General Electric Company, Philadelphia, PA	
T. J. McFall	1
General Electric Company, Pittsfield, MA	
R. W. Race	1
General Electric Company, Syracuse, NY	
J. L. Rogers	1
Dr. A. M. Vural	1
D. Winfield	1
Harris Scientific Services	
B. Harris	1
Honeywell Engineering Service Center	
C. Schmid	1
Hughes Aircraft Company, Buena Park, CA	
T. E. Posch	1
Hughes Aircraft Company, Fullerton, CA	
S. W. Autrey	1
IBM	
G. L. Demuth	1
Interstate Electronics Corporation	
R. O. Nielsen (Dept. 8011)	1
Kildare Corporation	
Dr. R. Mellen	1
Lincom Corporation	
Dr. T. A. Schonhoff	1
Magnavox Elec. Systems Company	
R. Kenefic (Dept. 525)	1
MSB Systems, Inc.	
A. Winder	1
Nichols Research Corporation	
T. L. Marzetta	1
ORI Company, Inc.	
G. Assard	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Orincon Corporation	
S. L. Marple	1
Prometheus, Inc., Newport, RI	
M. J. Barrett	1
Prometheus, Inc., Sharon, MA	
Dr. J. S. Byrnes	1
RAN Research Lab, Australia	1
Raytheon, Portsmouth, RI	
J. Bartram	1
R. Conner	1
S. S. Reese	1
RCA Corporation	
H. Urkowitz	1
Rockwell International	
L. T. Einstein	1
Dr. D. F. Elliott	1
SAIC, Falls Church, VA	
Dr. P. Mikhalevsky	1
SAIC, New London	
Dr. F. DiNapoli	1
SIMRAD SUBSEA A/S, Naval Systems Div.	
E. B. Lunde	1
Sperry Corporation, Defense Marine Systems Unit	1
Toyon Research Corporation	
M. L. Van Blaricum	1
Tracor	
Dr. T. J. Leih	1
TRW Federal Systems Group	
R. Prager	1
USEA S.P.A. La Spezia, Italy	
H. Van Asselt	1
Westinghouse Electric Corporation, Annapolis, MD	
H. Newman	1
Dr. H. L. Price	1
Westinghouse Electric Corporation, Waltham, MA	
D. Bennett	1
Bendat, Dr. J. S.	1
Bleistein, Dr. N.	1
Cohen, Dr. L.	1
Hahn, W. R.	1
Maltz, F.	1
Middleton, Dr. D.	1
Nash, H. E.	1
Papoutsanis, P.D.	1
Pohler, R. F.	1
Price, Dr. R.	1
Raisbeck, Dr. G.	1

INITIAL DISTRIBUTION LIST (CONT'D)

Addressee	No. of Copies
Richter, W.	1
Schulkin, Dr. M.	1
Urick, R. J.	1
Werbner, A.	1
Wilson, Dr. J. H.	1